

Bilayer Exciton insulator

Complete Electromagnetic responses and Meissner-like effects

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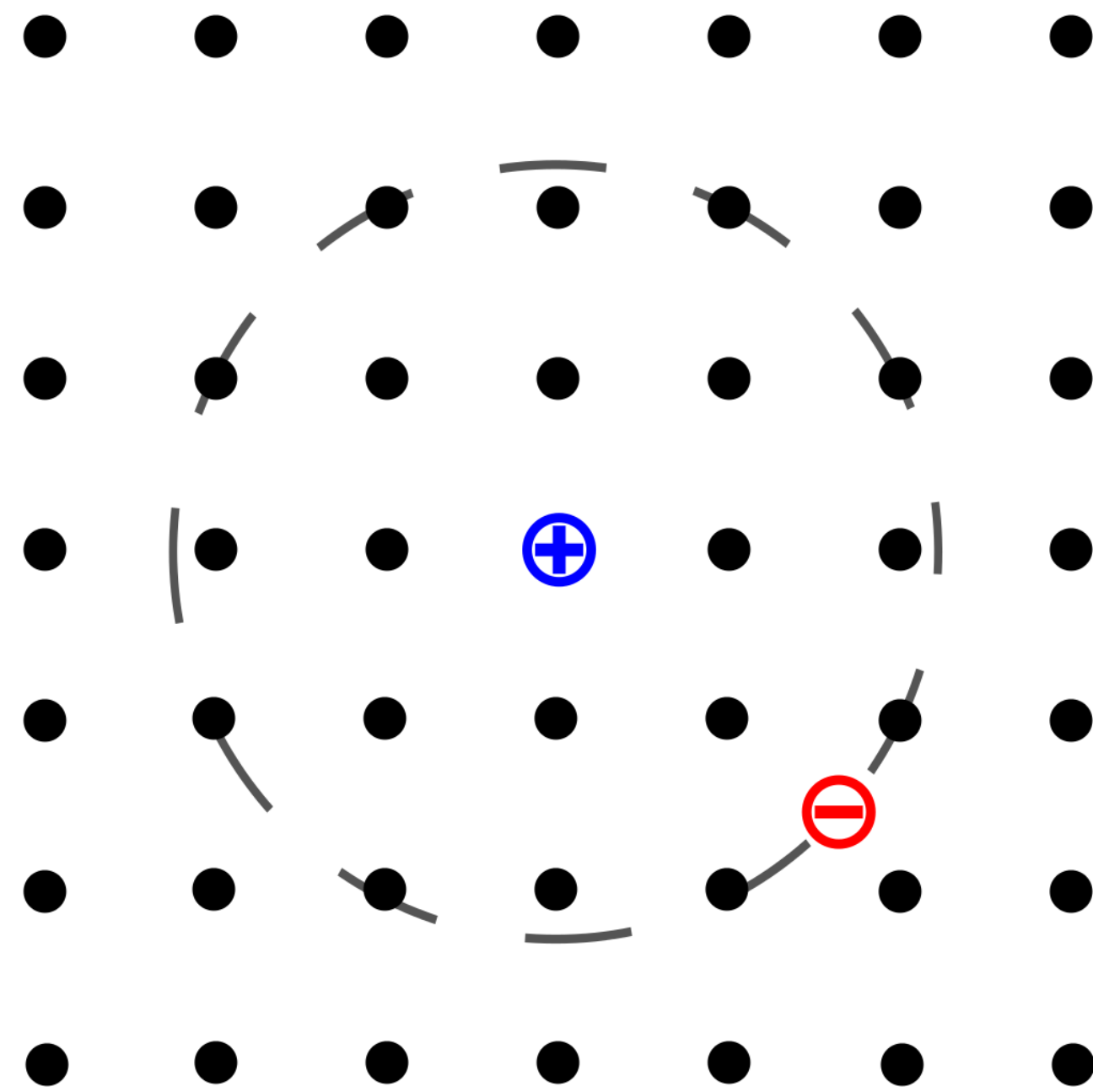
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& <https://doi.org/10.1103/PhysRevB.109.155107>

arXiv:2509.02142

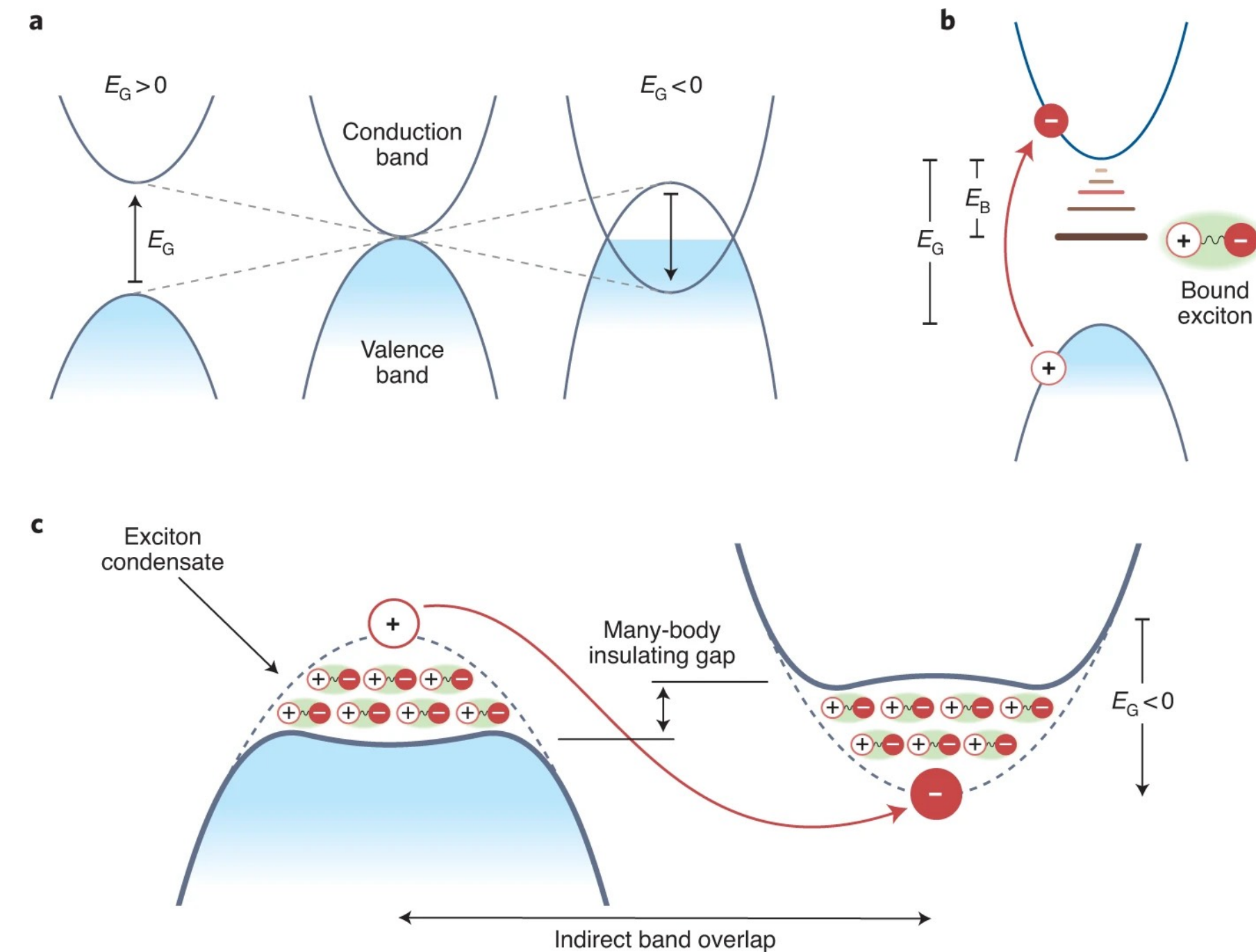
Excitons, Exciton insulators and Exciton (super)fluid

Insulator for the electron and holes but can form different states for excitons
i.e. fluid, superfluid or even crystals



BEC limit if $E_B > E_{\text{gap}}$

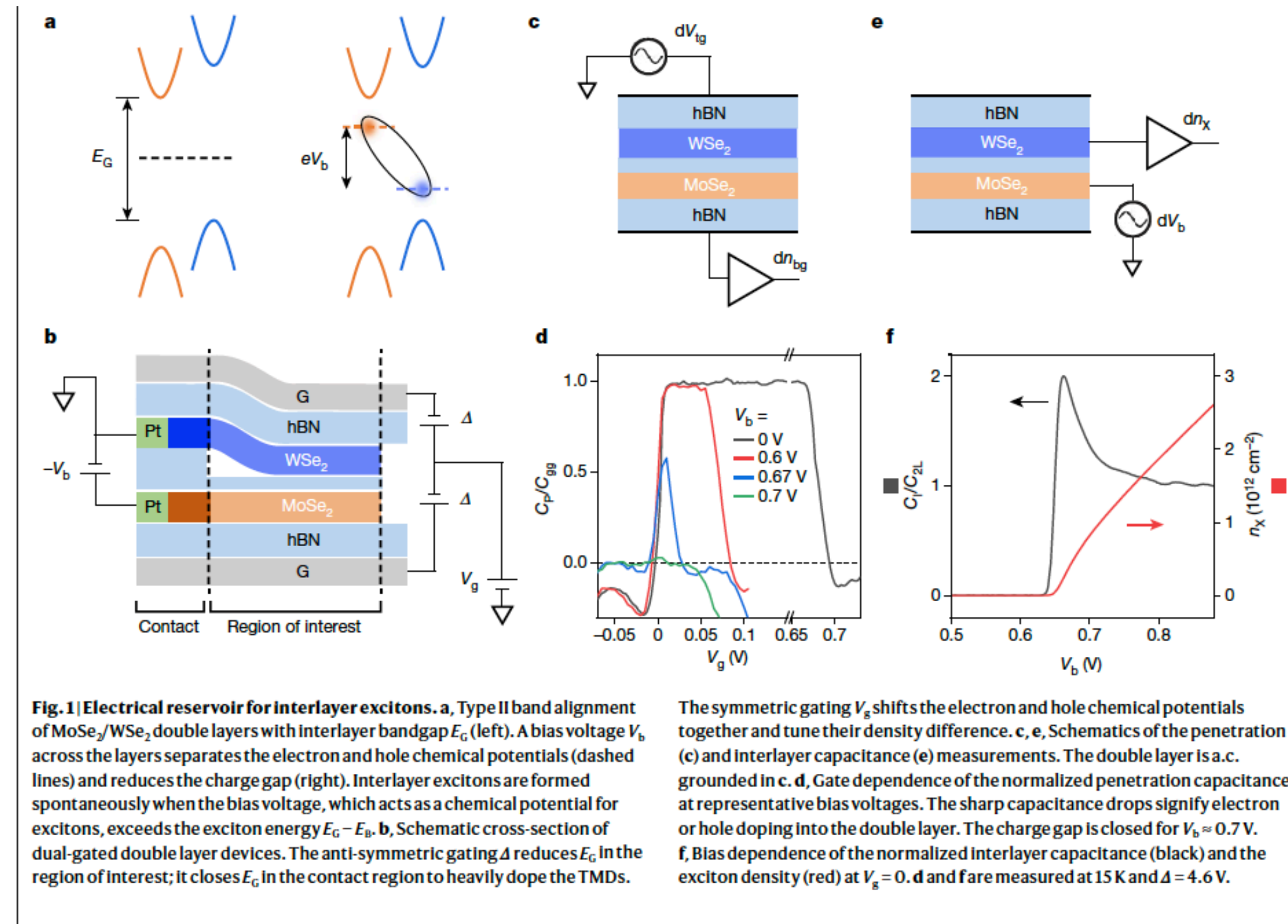
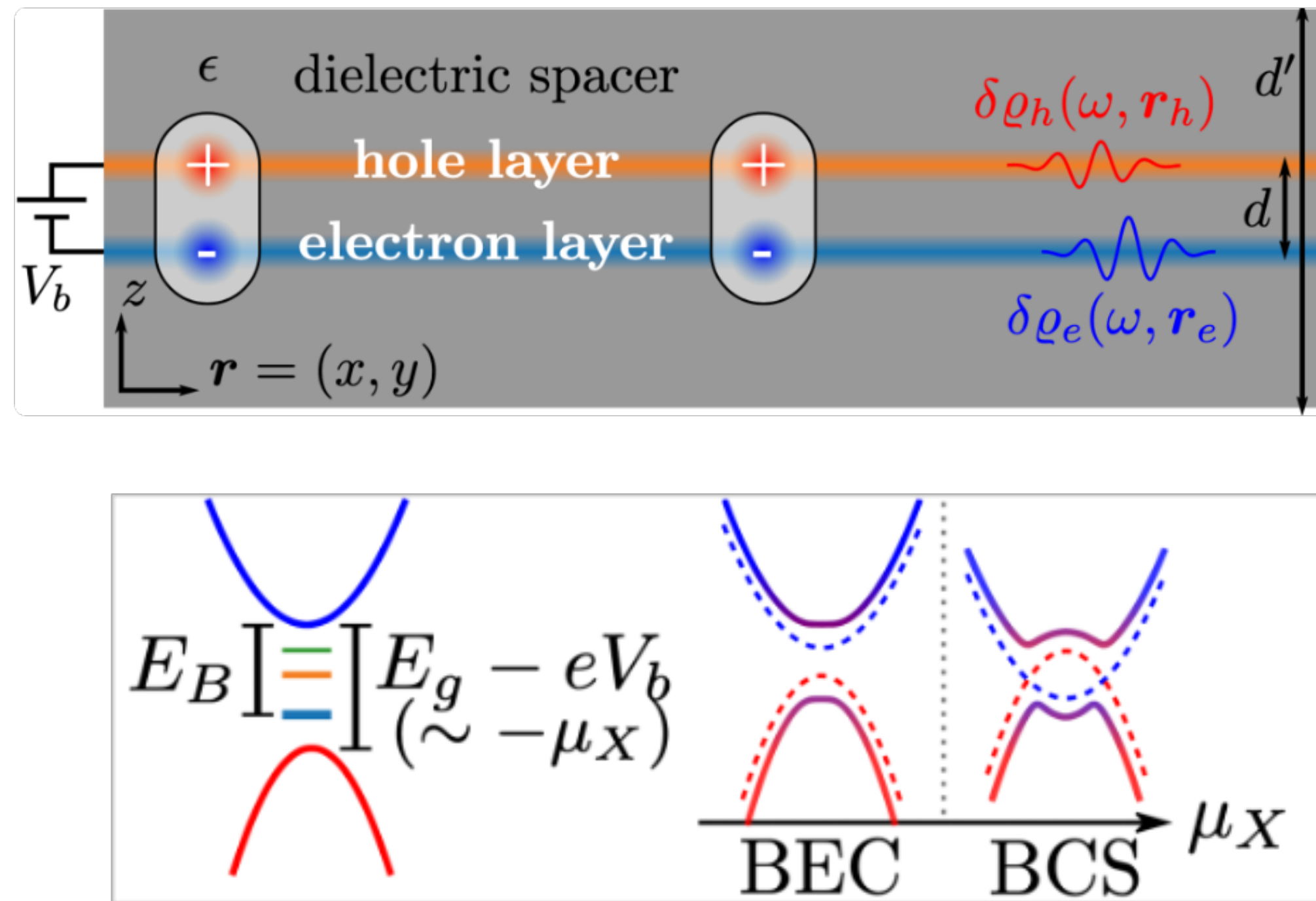
BCS limit



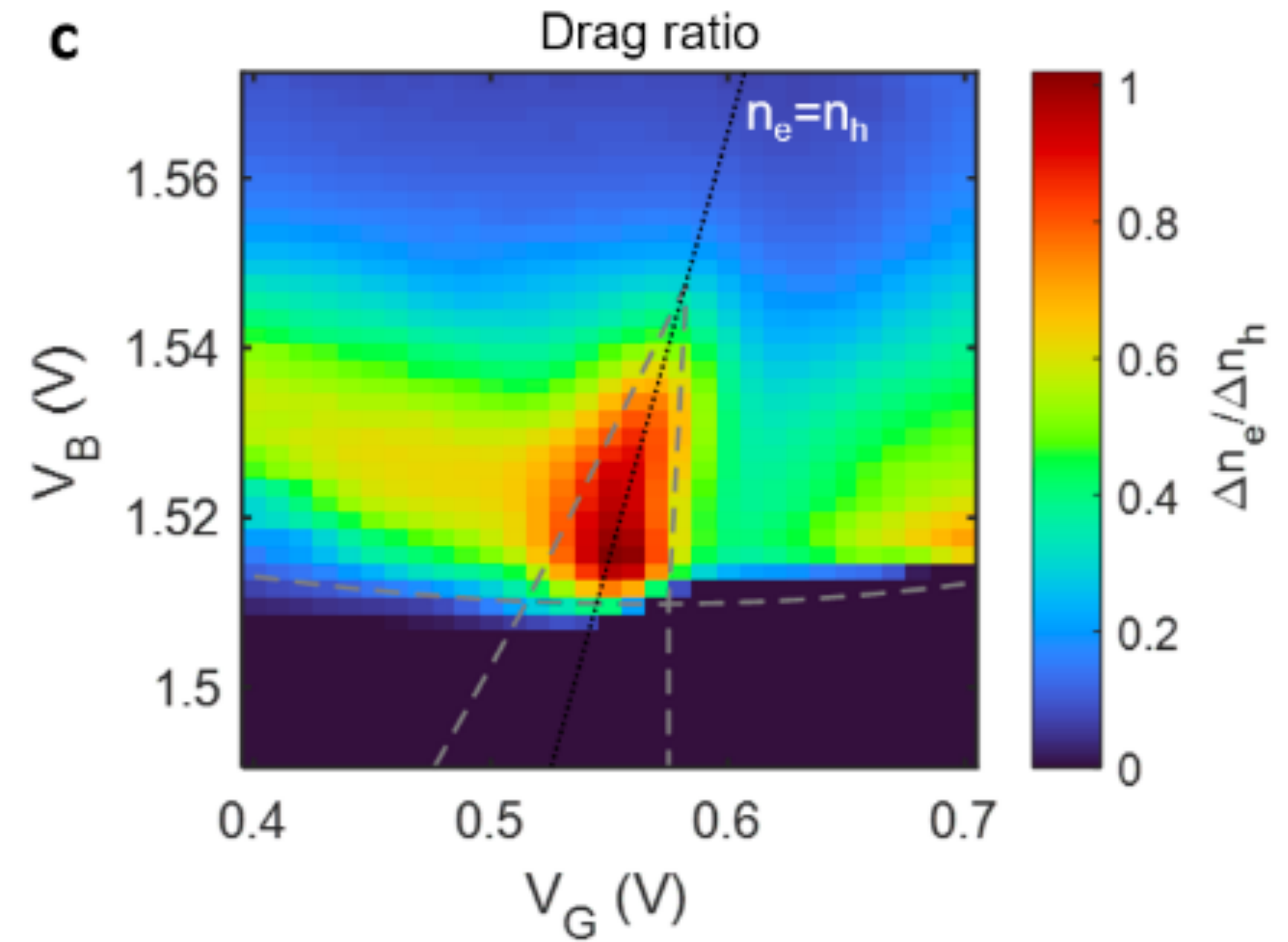
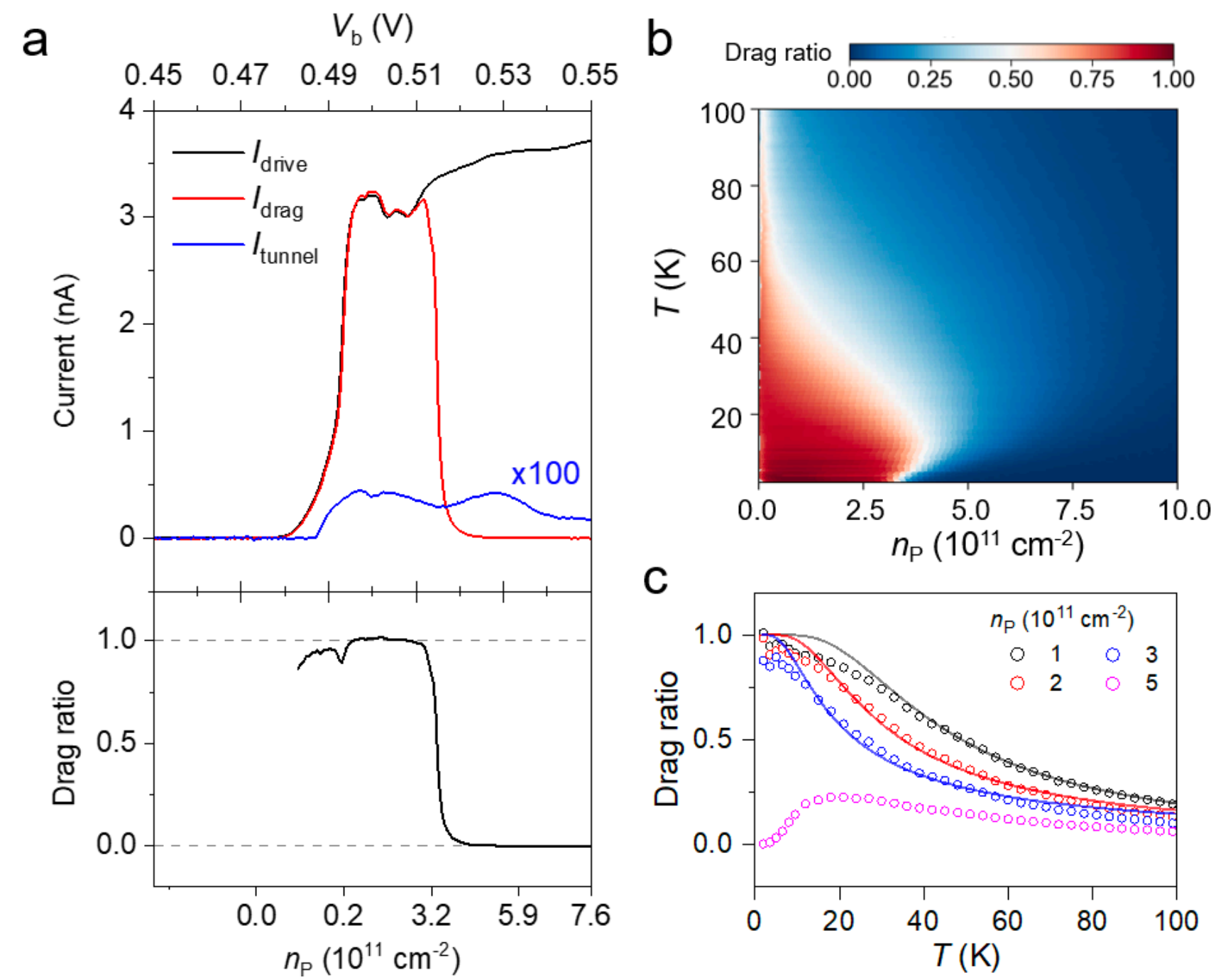
- [Vitor M. Pereira Nature Physics volume 18, 6–7 \(2022\)](#)

Inter-layer exciton systems

A new controllable platform to study EI



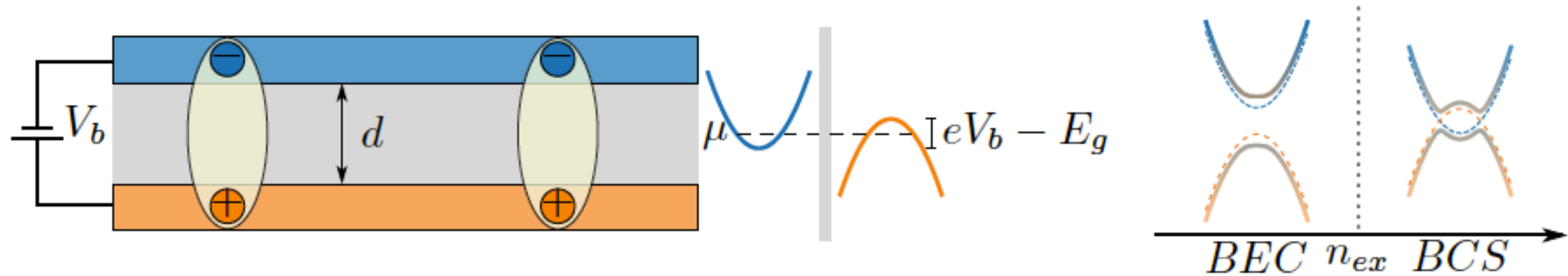
Nearly perfect drag



KinFai & Jie's Group in Cornell: SCIENCE, V388, 274, 2025

Feng Wang's Group in Berkeley: Science SCIENCE, V388, 278, 2025

Electron Hole Bilayer System



$$H = \sum_{s=eh\mathbf{k}} h_{ss'\mathbf{k}}^0 c_{s\mathbf{k}}^\dagger c_{s'\mathbf{k}} + \frac{1}{2\mathcal{V}} \sum_{ss'=eh\mathbf{k}_1\mathbf{k}_2\mathbf{q}} V_{ss'}(\mathbf{q}) c_{s\mathbf{k}_1}^\dagger c_{s'\mathbf{k}_2}^\dagger c_{s'\mathbf{k}_2+\mathbf{q}} c_{s\mathbf{k}_1-\mathbf{q}}$$

- Single particle Hamiltonian

$$h_{\mathbf{k}}^0 = \begin{bmatrix} \hbar^2 k^2 / 2m_e - \mu_{ex} & 0 \\ 0 & -\hbar^2 k^2 / 2m_h \end{bmatrix}$$

- Interaction

- Intra-layer: $V(r) = V_{s=s'}(r) = \frac{e^2}{\epsilon r} \Leftrightarrow V(q) = \frac{2\pi e^2}{\epsilon q}$
- Inter-layer: $U(r) = V_{s \neq s'}(r) = \frac{e^2}{\epsilon \sqrt{r^2 + d^2}} \Leftrightarrow U(q) = V(q) e^{-qd}$

Mean-field Hamiltonian

$$\hat{H}_{MF} = \sum_{\mathbf{k}} [c_{e\mathbf{k}}^\dagger, c_{h\mathbf{k}}^\dagger] \begin{bmatrix} \zeta_{\mathbf{k}} + \varepsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & \zeta_{\mathbf{k}} - \varepsilon_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{e\mathbf{k}} \\ c_{h\mathbf{k}} \end{bmatrix}$$

- Electron-hole asymmetry

$$\zeta_{\mathbf{k}} \equiv \hbar^2 k^2 \delta_m / 4m$$

- Renormalized band energy difference

$$\varepsilon_{\mathbf{k}} \equiv \frac{\hbar^2 k^2}{4m} - \frac{\mu_X}{2} + \frac{2\pi e^2 d n_X}{\epsilon} - \frac{1}{\mathcal{V}} \sum_{\mathbf{k}'} V(\mathbf{k} - \mathbf{k}') \rho_{ee\mathbf{k}'}^X$$

- Inter-layer coherence

$$\Delta_{\mathbf{k}} \equiv -\frac{1}{\mathcal{V}} \sum_{\mathbf{k}'} U(\mathbf{k} - \mathbf{k}') \rho_{eh\mathbf{k}'}^X$$

arg $\Delta_{\mathbf{k}}$ has a phase freedom
chosen to be real and negative

Excitonic units

- Reduced mass: $m = \frac{m_e m_h}{m_e + m_h} \approx 0.22 m_0$
- Dielectric constant: $\epsilon \approx 5$ (hBN)
 $\rightarrow \epsilon \approx 10$ (self-screening)
- Length unit: $a_B^* = \frac{\epsilon \hbar^2}{m^* e^2} \approx 2.4 \text{ nm}$
- Energy unit: $Ry^* = \frac{e^2}{2\epsilon a_B^*} \approx 30 \text{ meV}$

Independent parameters

- Exciton chemical potential: $\frac{\mu_{ex}}{Ry^*}$
- Interlayer distance: $\frac{d}{a_B^*}$
- Mass difference: $\delta_m \equiv \frac{m_e - m_h}{m_e + m_h}$

Single particle density matrix

$$\rho_{ss'\mathbf{k}}^X \equiv \langle c_{s'\mathbf{k}}^\dagger c_{s\mathbf{k}} \rangle$$

EM responses define a matter

What are they for inter-layer EI?

- Static E-field: Many-body Electric Breakdown VS single particle Zener tunnelling (PRX 14, 021047, 2024)
- Static: quantum oscillation of quantum capacitance under out of plane B-field (<https://doi.org/10.1103/PhysRevB.109.155107>)
- Dynamic: what are the responses generated by different collect modes?
- Dynamic: what is the gauge field that couples to exciton not electron or holes?
- Is there “Exciton Meissner effect”, “exciton Faraday effect” and “exciton Hall effect”? How to demonstrate them?

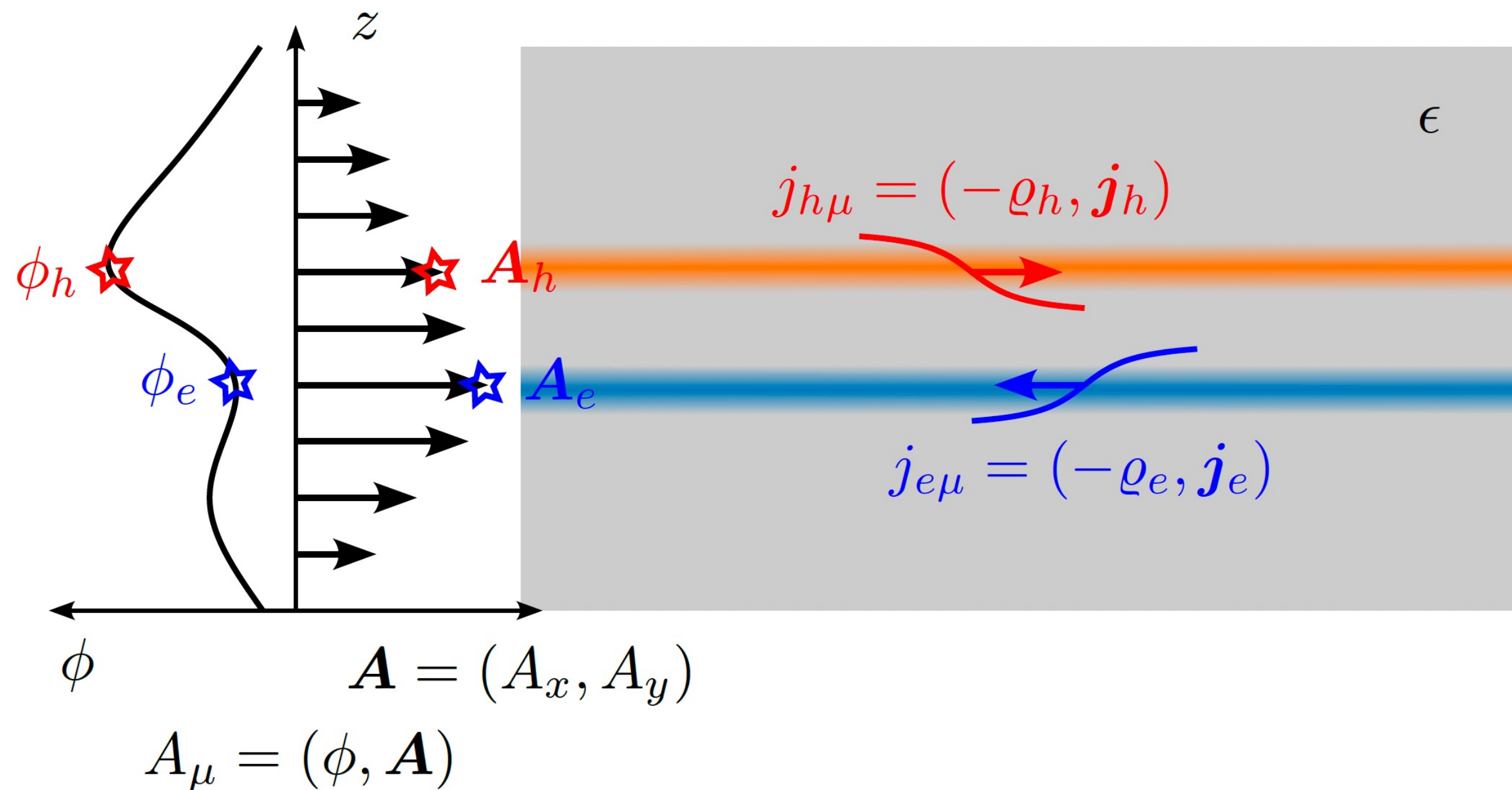
Dynamical response: collective modes

Coupling to the gauge field

- Coupling Hamiltonian

$$\hat{H}_c = \sum_{s=eh} \int d\mathbf{r} [\hat{\rho}_s(\mathbf{r})\phi_s(t, \mathbf{r}) - \hat{\mathbf{j}}_{s,p}(\mathbf{r}) \cdot \mathbf{A}_s(t, \mathbf{r})]$$

- $\hat{j}_{s\mu} = (-\hat{\rho}_s, \hat{\mathbf{j}}_{s,p})$: current operator in each layer
- $A_{s\mu} = (\phi_s, \mathbf{A}_s)$: gauge field in each layer



"Lorentz gauge"

$$\nabla \cdot \mathbf{A} + \partial_z A_z + \frac{\epsilon}{c^2} \partial_t \phi = 0$$

$$\left(\frac{\epsilon}{c^2} \partial_t^2 - \nabla^2 - \partial_z^2 \right) \begin{bmatrix} \phi(t, \mathbf{r}, z) \\ A_z(t, \mathbf{r}, z) \\ \mathbf{A}(t, \mathbf{r}, z) \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} 4\pi \rho(t, \mathbf{r}, z) \\ 0 \\ 4\pi \epsilon \mathbf{j}(t, \mathbf{r}, z)/c^2 \end{bmatrix}$$

$$A_z = 0$$

$$\nabla \cdot \mathbf{A} + \frac{\epsilon}{c^2} \partial_t \phi = 0$$

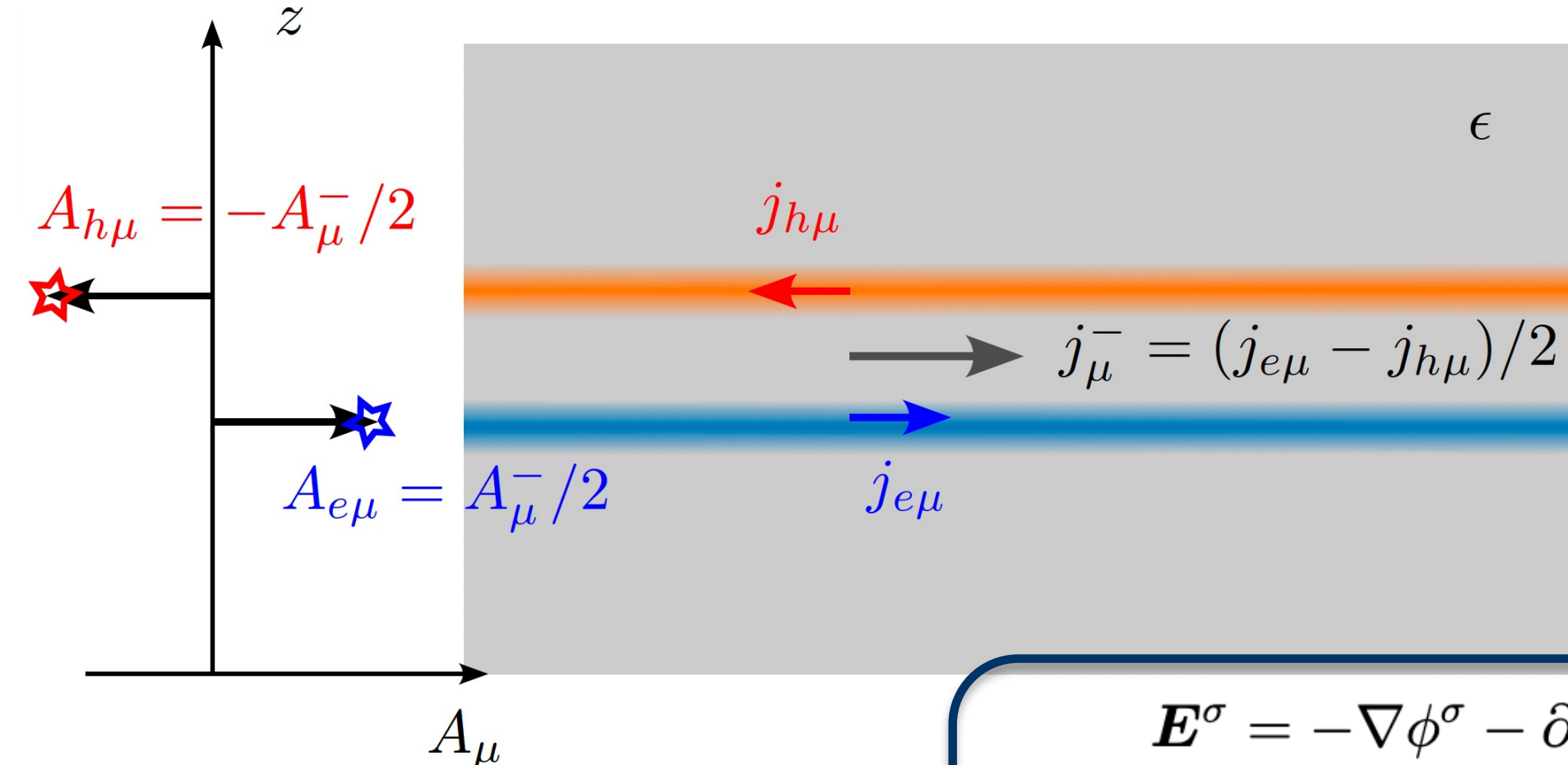
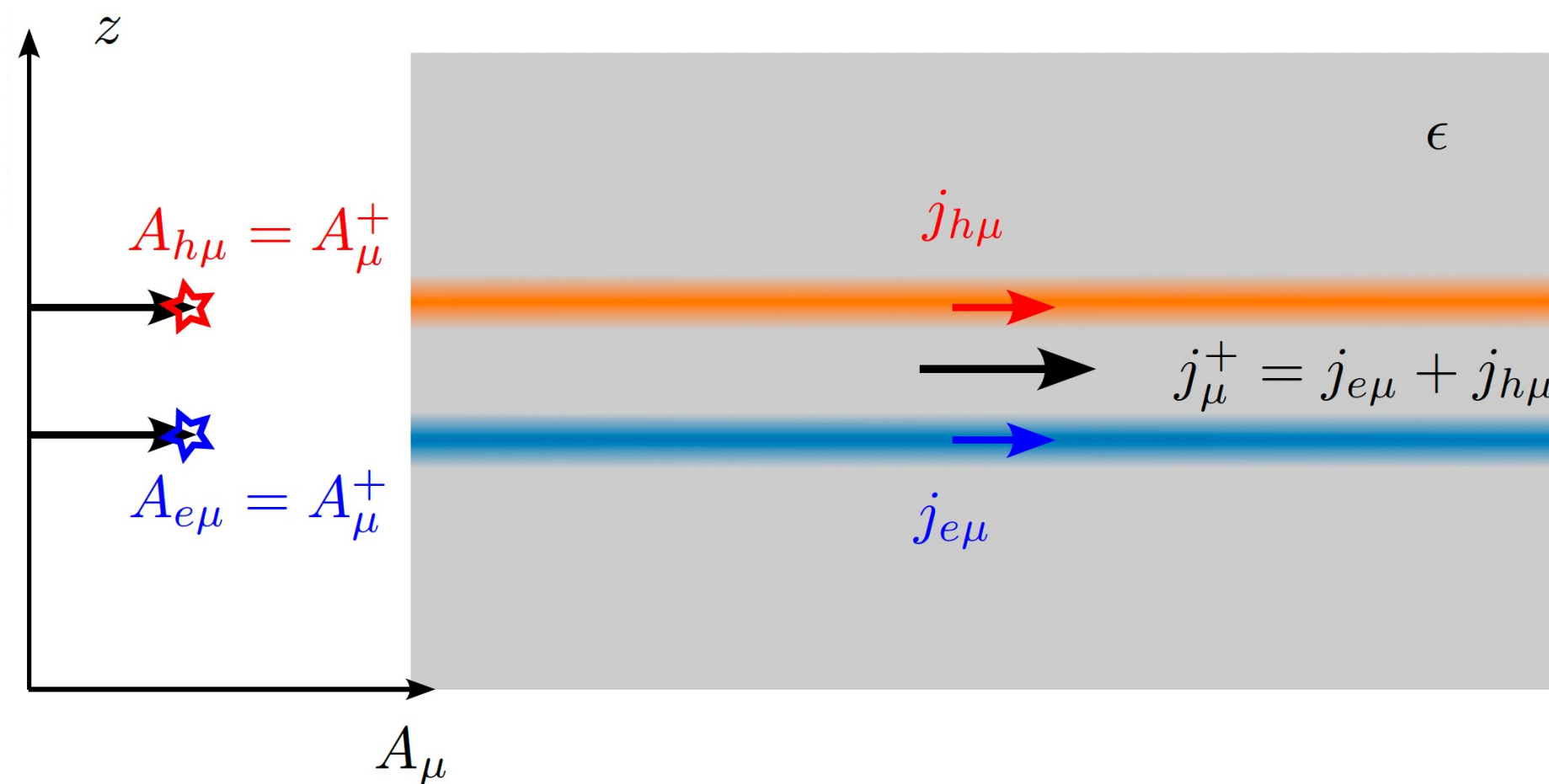
1 Electromagnetic response functions

Coupling to the gauge field

• Coupling Hamiltonian

$$\hat{H}_c = \sum_{s=e,h} \int d\mathbf{r} [\hat{\rho}_s(\mathbf{r})\phi_s(t, \mathbf{r}) - \hat{\mathbf{j}}_{s,p}(\mathbf{r}) \cdot \mathbf{A}_s(t, \mathbf{r})]$$

- $\hat{j}_{s\mu} = (-\hat{Q}_s, \hat{j}_{s,p})$: current operator in each layer
- $A_{s\mu} = (\phi_s, \mathbf{A}_s)$: gauge field in each layer



$$\hat{H}_c = \sum_{\sigma=\pm} \int d\mathbf{r} [\hat{\rho}^{\sigma}(\mathbf{r})\phi^{\sigma}(t, \mathbf{r}) - \hat{\mathbf{j}}_p^{\sigma}(\mathbf{r}) \cdot \mathbf{A}^{\sigma}(t, \mathbf{r})]$$

$$\begin{aligned} \mathbf{E}^{\sigma} &= -\nabla\phi^{\sigma} - \partial_t\mathbf{A}^{\sigma} \\ E_z &= -\partial_z\phi \approx -(\phi^h - \phi^e)/d = \phi^-/d \\ \mathbf{B} &= (-\partial_z A_y, \partial_z A_x) \approx \mathbf{A}^- \times \hat{z}/d \\ B_z^{\sigma} &= \nabla \times \mathbf{A}^{\sigma} \end{aligned}$$

Coupling to the gauge field

• Coupling Hamiltonian

$$\hat{H}_c = \sum_{s=eh} \int d\mathbf{r} [\hat{\rho}_s(\mathbf{r})\phi_s(t, \mathbf{r}) - \hat{\mathbf{j}}_{s,p}(\mathbf{r}) \cdot \mathbf{A}_s(t, \mathbf{r})]$$

- $\hat{j}_{s\mu} = (-\hat{\rho}_s, \hat{j}_{s,p})$: current operator in each layer
- $A_{s\mu} = (\phi_s, \mathbf{A}_s)$: gauge field in each layer

Layer symmetric gauge field & charge density (current) operators

$$A_{\mu}^{+}(t, \mathbf{r}) \equiv [A_{e\mu}(t, \mathbf{r}) + A_{h\mu}(t, \mathbf{r})]/2$$

$$\hat{\rho}^{+}(\mathbf{r}) = -e\Psi^{\dagger}(\mathbf{r})\Psi(\mathbf{r}),$$

$$\hat{\mathbf{j}}_p^{+}(\mathbf{r}) = \frac{ie\hbar}{4m}\Psi^{\dagger}(\mathbf{r})\sigma_z\nabla_{\mathbf{r}}\Psi(\mathbf{r}) + h.c..$$

$$\hat{\mathbf{j}}_d^{+}(t, \mathbf{r}) = -\frac{e^2}{2m}\Psi^{\dagger}(\mathbf{r})\sigma_z\Psi(\mathbf{r})\mathbf{A}^{+}(t, \mathbf{r})$$

Layer anti-symmetric gauge field & exciton density (current) operators

$$A_{\mu}^{-}(t, \mathbf{r}) \equiv A_{e\mu}(t, \mathbf{r}) - A_{h\mu}(t, \mathbf{r})$$

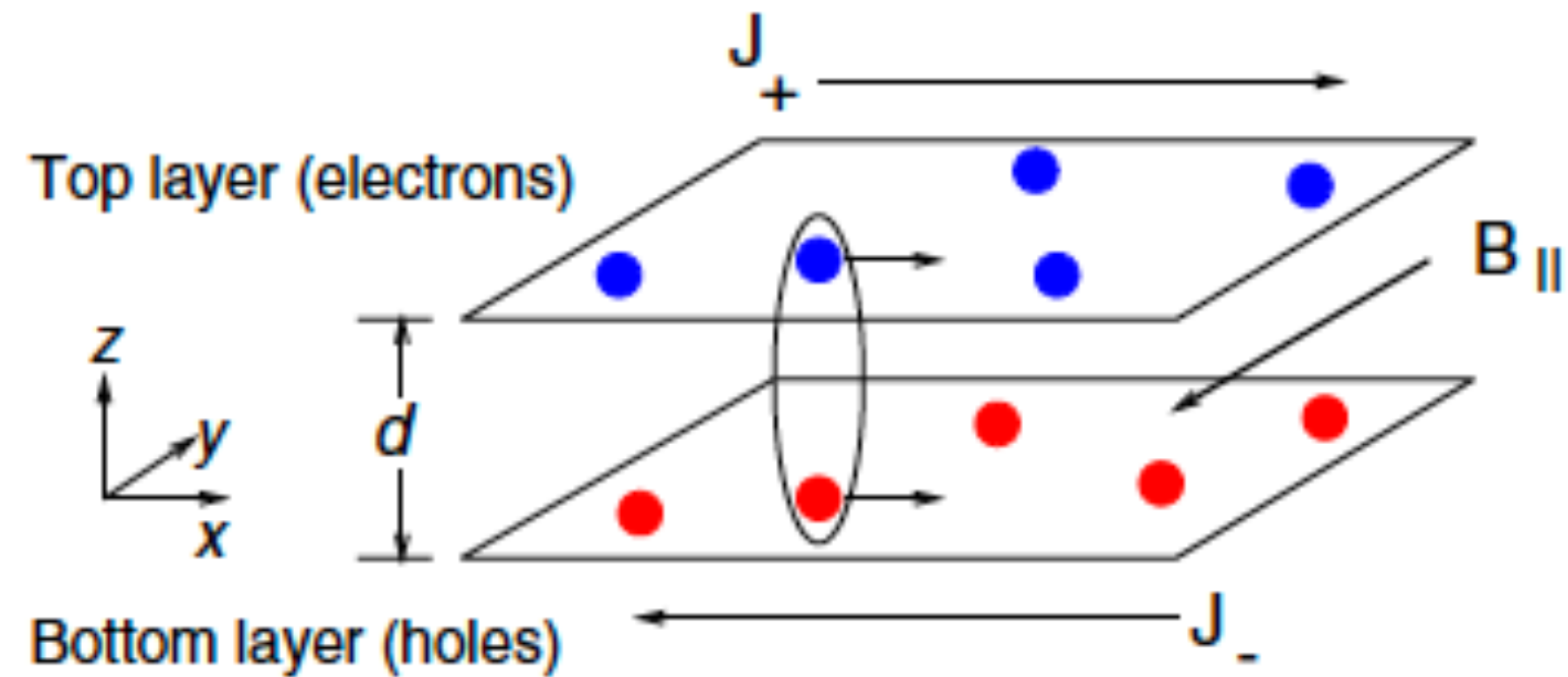
$$\hat{\rho}^{-}(\mathbf{r}) = -\frac{e}{2}\Psi^{\dagger}(\mathbf{r})\sigma_z\Psi(\mathbf{r}),$$

$$\hat{\mathbf{j}}_p^{-}(\mathbf{r}) = \frac{ie\hbar}{8m}\Psi^{\dagger}(\mathbf{r})\nabla_{\mathbf{r}}\Psi(\mathbf{r}) + h.c..$$

$$\hat{\mathbf{j}}_d^{-}(t, \mathbf{r}) = -\frac{e^2}{8m}\Psi^{\dagger}(\mathbf{r})\sigma_z\Psi(\mathbf{r})\mathbf{A}^{-}(t, \mathbf{r})$$

$$\hat{H}_c = \sum_{\sigma=\pm} \int d\mathbf{r} [\hat{\rho}^{\sigma}(\mathbf{r})\phi^{\sigma}(t, \mathbf{r}) - \hat{\mathbf{j}}_p^{\sigma}(\mathbf{r}) \cdot \mathbf{A}^{\sigma}(t, \mathbf{r})]$$

Exciton Meissner effect causes the in plane B field to couple to exciton current



$$E_z \approx -\frac{\phi_h - \phi_e}{d} = \frac{\phi^-}{d} \quad \mathbf{E}_t^- = -\vec{\nabla}\phi^- - \frac{\partial \mathbf{A}_t^-}{\partial t} = d\vec{\nabla}E_z - \frac{\partial \mathbf{A}_t^-}{\partial t}$$

$$\mathbf{B}_t^+ \approx \frac{1}{d}\mathbf{A}_t^- \times \hat{e}_z \quad B_z^- \approx \vec{\nabla} \times \mathbf{A}_t^- \quad \vec{\nabla} \cdot \mathbf{B}_t^+ = -dB_z^- = -d\vec{\nabla} \times \mathbf{A}_t^- = -db_{exciton}^z$$

FIG. 1 (color online). Schematic bilayer electron-hole system. The electrons and holes form bound excitons which condense at low densities. We predict that an in-plane field $\mathbf{B}_{||}$ will produce equal but opposite currents $\mathbf{J}_+$ and $\mathbf{J}_-$ in the two layers.

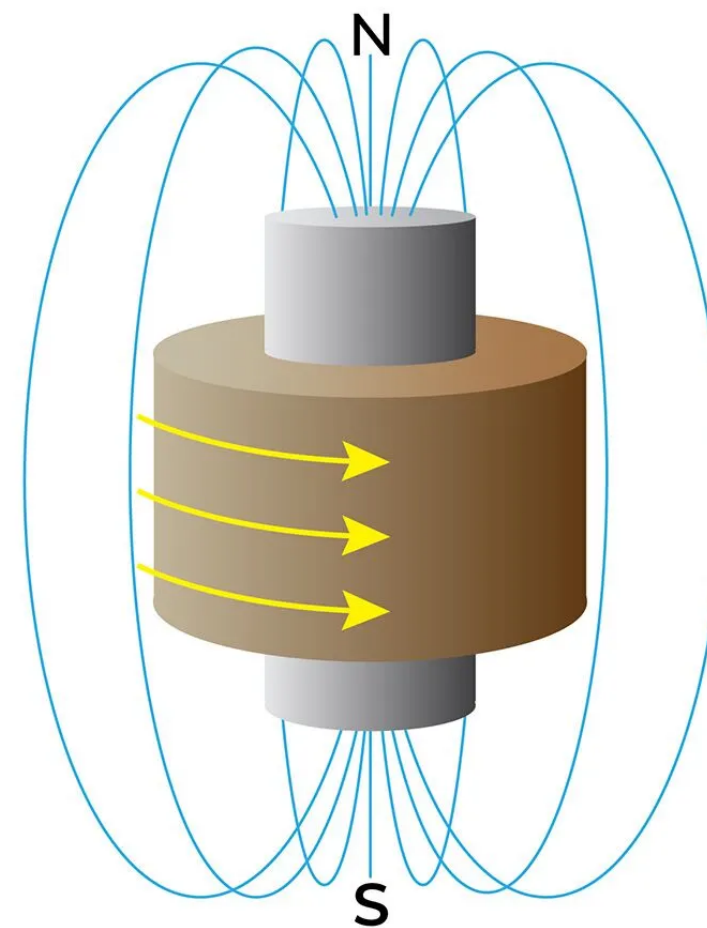
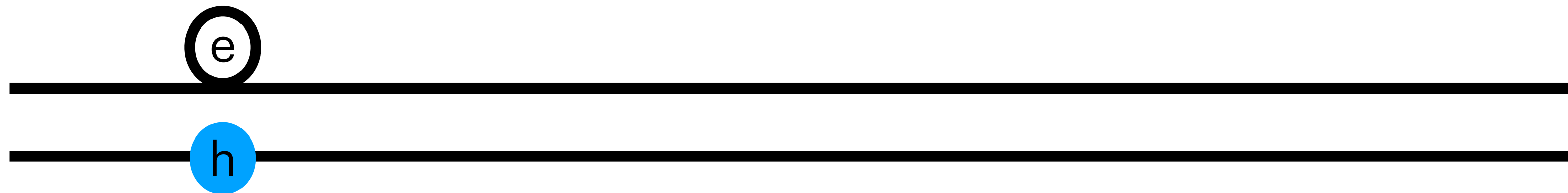
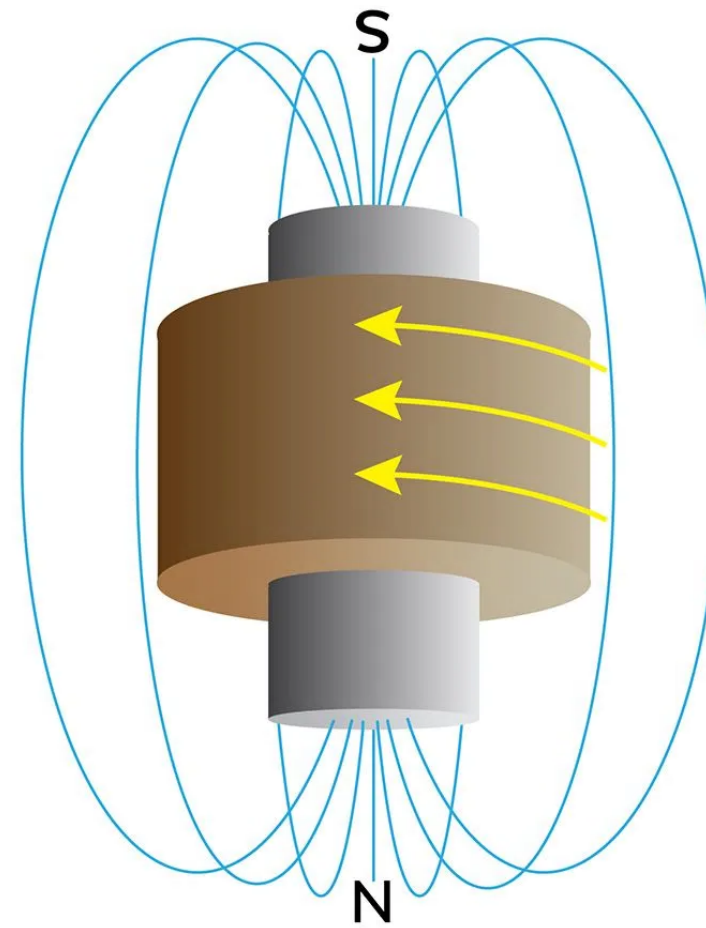
Ginzburg-Landau theory

$$\mathbf{j}_X = \frac{e^2 \rho_X}{m} \mathbf{B}_t \times \hat{e}_z$$

Exciton London equation

$$\frac{1}{2m^*} \Psi^* (-i\hbar \vec{\nabla} - \frac{e}{c} \mathbf{B}_t \times \hat{e}_z)^2 \Psi + \alpha \Psi^* \Psi + \beta |\Psi|^4$$

Setup for generating vortex lattice of exciton superfluid



Maxwell equation of the bilayer system

- "Lorentz gauge" in the dielectric environment

$$\nabla \cdot \mathbf{A} + \partial_z A_z + \frac{1}{c^2} \partial_t \phi = 0$$

$$c = 1/\sqrt{\epsilon_0 \mu_0 \epsilon \mu}$$

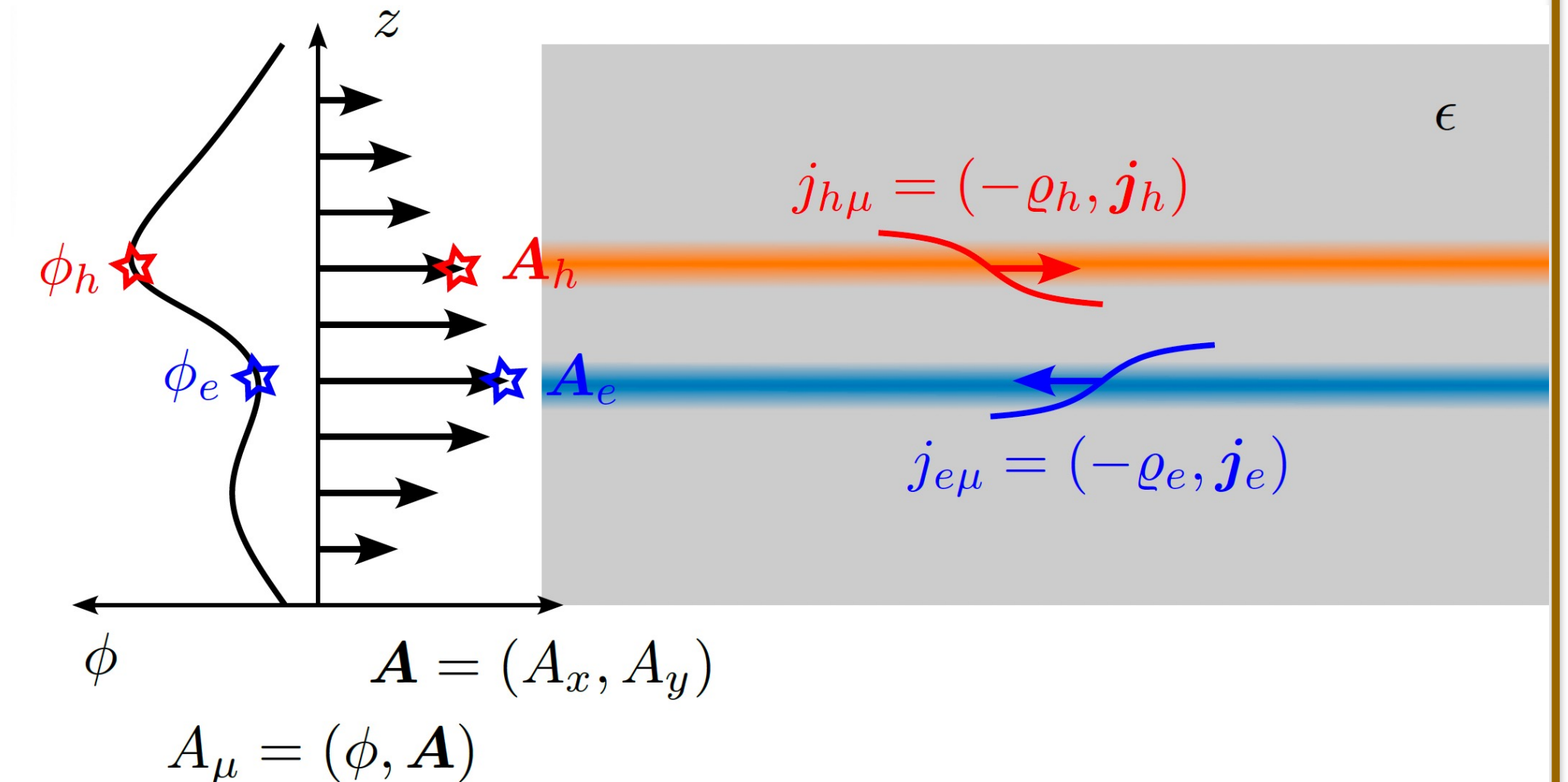
- d'Alembert equation

$$\left(\frac{1}{c^2} \partial_t^2 - \nabla^2 - \partial_z^2 \right) \begin{bmatrix} \phi(t, \mathbf{r}, z) \\ A_z(t, \mathbf{r}, z) \\ \mathbf{A}(t, \mathbf{r}, z) \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} 4\pi \rho(t, \mathbf{r}, z) \\ 0 \\ 4\pi \mathbf{j}(t, \mathbf{r}, z)/c^2 \end{bmatrix}$$

- Bound current

$$\begin{aligned} j_{\mu=0,1,2}(\omega, \mathbf{q}, z) &= j_{e\mu}(\omega, \mathbf{q}) \delta(z + z_0) + j_{h\mu}(\omega, \mathbf{q}) \delta(z - z_0) \\ &= j_{\mu}^+(\omega, \mathbf{q}) [\delta(z - z_0) + \delta(z + z_0)] + \frac{j_{\mu}^-(\omega, \mathbf{q})}{2} [\delta(z - z_0) - \delta(z + z_0)] \end{aligned}$$

$$\begin{aligned} j_{\mu}^+ &= \frac{1}{2} (j_{e\mu} + j_{h\mu}) \\ j_{\mu}^- &= j_{e\mu} - j_{h\mu} \end{aligned}$$



Maxwell equation of the bilayer system

- Evanescent solution

$$A_\mu(\omega, \mathbf{q}, z) = A_\mu^+(\omega, \mathbf{q}) \frac{e^{-\lambda|z-z_0|} + e^{-\lambda|z+z_0|}}{1 + e^{-2\lambda z_0}} - A_\mu^-(\omega, \mathbf{q}) \frac{e^{-\lambda|z-z_0|} - e^{-\lambda|z+z_0|}}{2(1 - e^{-2\lambda z_0})}$$

$$A_z = 0$$

$$-\omega^2/c^2 + q^2 - \lambda^2 = 0$$

$$A_\mu^+(\omega, \mathbf{q}) = \frac{1}{2}[A_\mu(\omega, \mathbf{q}, z_0) + A_\mu(\omega, \mathbf{q}, -z_0)]$$

$$A_\mu^-(\omega, \mathbf{q}) = A_\mu(\omega, \mathbf{q}, -z_0) - A_\mu(\omega, \mathbf{q}, z_0)$$

- 2D Maxwell equation

$$\frac{4\lambda}{1 + e^{-2\lambda z_0}} \phi^+ = -\frac{4\pi}{\epsilon} \rho^+$$

$$\frac{4\lambda}{1 + e^{-2\lambda z_0}} A_i^+ = \frac{4\pi}{\epsilon c^2} j_i^+$$

$$\frac{\lambda}{1 - e^{-2\lambda z_0}} \phi^- = \frac{4\pi}{\epsilon} \rho^-$$

$$\frac{\lambda}{1 - e^{-2\lambda z_0}} A_i^- = -\frac{4\pi}{\epsilon c^2} j_i^-$$

Response function

$$j_\mu^\sigma(\omega, \mathbf{q}) = \sum_\nu K_{\mu\nu}^\sigma(\omega, \mathbf{q}) A_\nu^\sigma(\omega, \mathbf{q})$$

- State of matter

$$j_\mu^\sigma(\omega, \mathbf{q}) = \sum_\nu K_{\mu\nu}^{0,\sigma}(\omega, \mathbf{q}) A_\nu^\sigma(\omega, \mathbf{q})$$

Response functions

- Linear response relation

$$j_{\mu}^{\sigma}(\omega, \mathbf{q}) = \sum_{\nu} K_{\mu\nu}^{\sigma}(\omega, \mathbf{q}) A_{\nu}^{\sigma}(\omega, \mathbf{q})$$

- Ward identity (charge conservation)

$$\sum_{\mu} q_{\mu} K_{\mu\nu}^{\sigma}(\omega, \mathbf{q}) = \sum_{\nu} K_{\mu\nu}^{\sigma}(\omega, \mathbf{q}) q_{\nu} = 0$$

- Rotation symmetry

$$K_{ab}^{\sigma}(\omega, \mathbf{q}) = K_L^{\sigma}(\omega, \mathbf{q}) \frac{q_a q_b}{|\mathbf{q}|^2} + K_T^{\sigma}(\omega, \mathbf{q}) \left(\delta_{ab} - \frac{q_a q_b}{|\mathbf{q}|^2} \right)$$

$$K_{\mu\nu}^{\sigma}(\omega, \mathbf{q} = (q, 0)) = \begin{bmatrix} \frac{q^2}{\omega^2} K_L^{\sigma} & -\frac{q}{\omega} K_L^{\sigma} & 0 \\ -\frac{q}{\omega} K_L^{\sigma} & K_L^{\sigma} & 0 \\ 0 & 0 & K_T^{\sigma} \end{bmatrix}$$

only 2 independent components in the 9 responses functions

$$\sigma = + \text{ for charge}$$

$$K_{\mu\nu}^{+}(\omega, \mathbf{q}) = - (1 - \delta_{\mu 0}) \delta_{\mu\nu} \frac{e^2 n_X}{m} - C_{\hat{j}_{p\mu}^{+} \hat{j}_{p\nu}^{+}}(\omega, \mathbf{q})$$

$$\sigma = - \text{ for exciton}$$

$$K_{\mu\nu}^{-}(\omega, \mathbf{q}) = - (1 - \delta_{\mu 0}) \delta_{\mu\nu} \frac{e^2 n_X}{4m} - C_{\hat{j}_{p\mu}^{-} \hat{j}_{p\nu}^{-}}(\omega, \mathbf{q})$$

Dynamics Equation Approach

- Dynamic equation

$$\rho_{ij\mathbf{k}}(t, \mathbf{q}) \equiv \langle c_{j\mathbf{k}-\mathbf{q}/2}^\dagger c_{i\mathbf{k}+\mathbf{q}/2} \rangle$$

$$i\hbar\partial_t\rho_{ij\mathbf{k}}(t, \mathbf{q}) = \langle [c_{j\mathbf{k}-\mathbf{q}/2}^\dagger c_{i\mathbf{k}+\mathbf{q}/2}, \hat{H} + \hat{H}_c] \rangle$$

Coupling Hamiltonian

$$\hat{H}_c = \frac{1}{\mathcal{V}} \sum_{\mathbf{q}} f(t, \mathbf{q}) \hat{O}(-\mathbf{q})$$

$$\hat{O}(\mathbf{q}) = \sum_{i,j=c,v} \sum_{\mathbf{k}} o_{ij\mathbf{k}}(\mathbf{q}) c_{i\mathbf{k}-\mathbf{q}/2}^\dagger c_{j\mathbf{k}+\mathbf{q}/2}$$

- Linearization

$$\rho_{ij\mathbf{k}}(t, \mathbf{q}) = \sum \rho_{ij\mathbf{k}}^{(n)}(t, \mathbf{q})$$

$\rho^{(n)}$: n-th order of f

$$\rho_{ij\mathbf{k}}^{(0)}(t, \mathbf{q}) = \rho_{ij\mathbf{k}}^X \delta_{\mathbf{q}0} = \delta_{ij} \delta_{iv} \delta_{\mathbf{q}0}$$

$$i\hbar\tau_z\partial_t \begin{bmatrix} \rho_{cv\mathbf{k}}^{(1)}(\mathbf{q}) \\ \rho_{vc-\mathbf{k}}^{(1)}(\mathbf{q}) \end{bmatrix} = \sum_{\mathbf{k}'} \mathcal{H}_{\mathbf{k},\mathbf{k}'}(\mathbf{q}) \begin{bmatrix} \rho_{cv\mathbf{k}'}^{(1)}(\mathbf{q}) \\ \rho_{vc-\mathbf{k}'}^{(1)}(\mathbf{q}) \end{bmatrix} + \frac{1}{\mathcal{V}} \begin{bmatrix} o_{cv\mathbf{k}}(-\mathbf{q}) \\ o_{vc-\mathbf{k}}(-\mathbf{q}) \end{bmatrix} f(t, \mathbf{q})$$

- Formal solution in frequency domain

$$\begin{bmatrix} \rho_{cv\mathbf{k}}^{(1)}(\omega, \mathbf{q}) \\ \rho_{vc-\mathbf{k}}^{(1)}(\omega, \mathbf{q}) \end{bmatrix} = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}'} [\hbar\omega^+ \tau_z - \mathcal{H}(\mathbf{q})]_{\mathbf{k},\mathbf{k}'}^{-1} \begin{bmatrix} o_{cv\mathbf{k}'}(-\mathbf{q}) \\ o_{vc-\mathbf{k}'}(-\mathbf{q}) \end{bmatrix} f(\omega, \mathbf{q})$$

Dynamics Equation Approach

- Collective modes and excitation energies

$$\sum_{\mathbf{k}'} \mathcal{H}_{\mathbf{k},\mathbf{k}'}(\mathbf{q}) \Phi_{n\mathbf{k}'}(\mathbf{q}) = \hbar\omega_n(\mathbf{q}) \tau_z \Phi_{n\mathbf{k}}(\mathbf{q})$$

Generalized eigenvalue equation

$$\Phi_m^\dagger(\mathbf{q}) \mathcal{H}(\mathbf{q}) \Phi_n(\mathbf{q}) = \delta_{mn}, \quad \Phi_m^\dagger(\mathbf{q}) \tau_z \Phi_n(\mathbf{q}) = \frac{\delta_{mn}}{\hbar\omega_n(\mathbf{q})}$$

Generalized orthonormal relation

- Solution expressed by collective modes

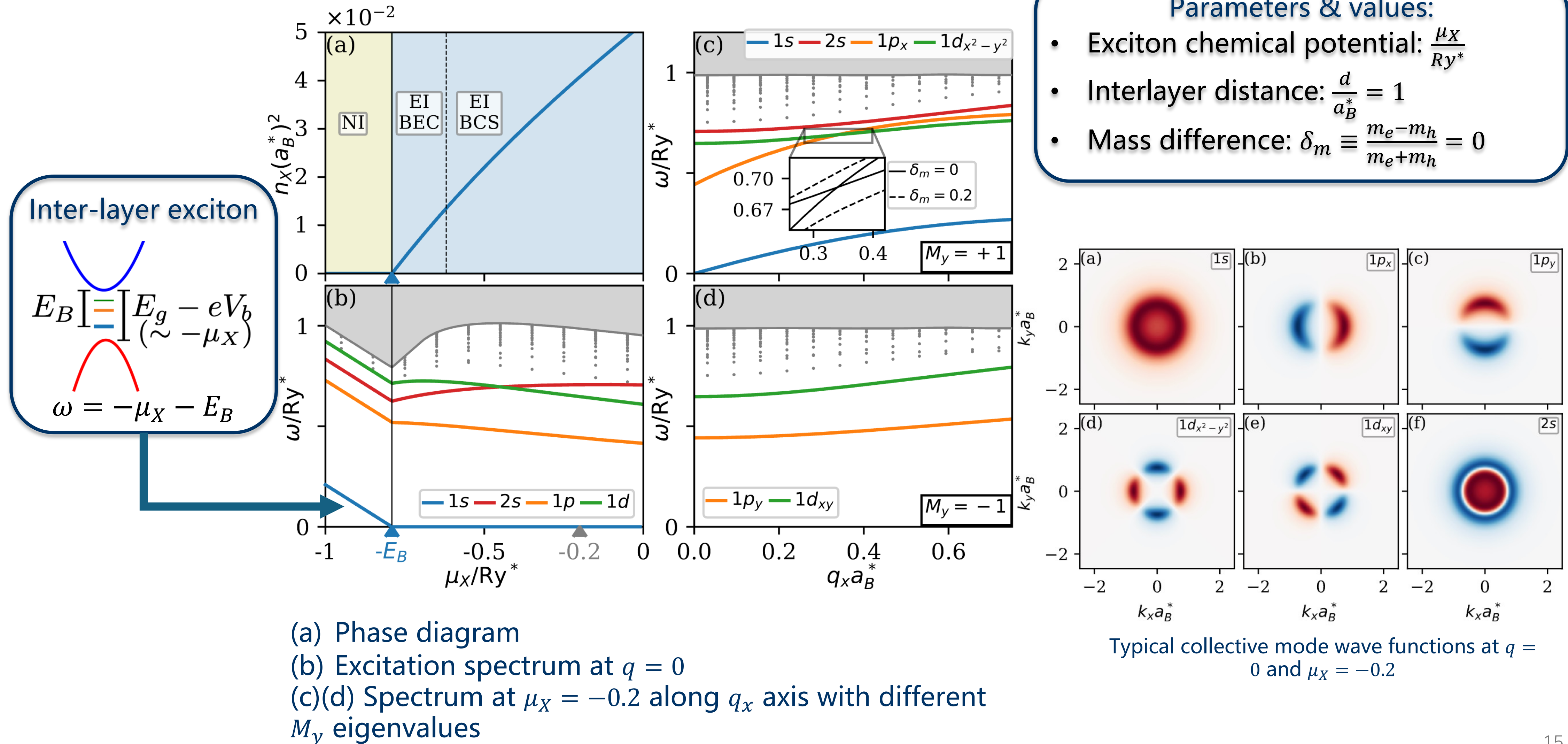
$$\begin{bmatrix} \rho_{cv\mathbf{k}}^{(1)}(\omega, \mathbf{q}) \\ \rho_{vc-\mathbf{k}}^{(1)}(\omega, \mathbf{q}) \end{bmatrix} = \frac{1}{\mathcal{V}} \sum_n \frac{\omega_n(\mathbf{q}) \Phi_{n\mathbf{k}}(\mathbf{q}) O_n(\mathbf{q})}{\omega^+ - \omega_n(\mathbf{q})} f(\omega, \mathbf{q}) \quad O_n(\mathbf{q}) \equiv \sum_{\mathbf{k}} \Phi_{n\mathbf{k}}^\dagger(\mathbf{q}) \begin{bmatrix} o_{cv\mathbf{k}}(-\mathbf{q}) \\ o_{vc-\mathbf{k}}(-\mathbf{q}) \end{bmatrix}$$

- Correlation functions

$$\langle \hat{O}' \rangle(\omega, \mathbf{q}) = \sum_{ijk} o'_{ijk}(\mathbf{q}) \rho_{jik}^{(1)}(\omega, \mathbf{q}) = C_{\hat{O}', \hat{O}}(\omega, \mathbf{q}) f(\omega, \mathbf{q})$$

$$C_{\hat{O}', \hat{O}}(\omega, \mathbf{q}) = \frac{1}{\mathcal{V}} \sum_n \frac{\omega_n(\mathbf{q}) [O'_n(\mathbf{q})]^* O_n(\mathbf{q})}{\omega + i\eta - \omega_n(\mathbf{q})}$$

2 The collective modes spectrum



Also see ZhiYuan Sun & Andy Millis, PRB 101, 224305 (2020)

Dispersion of the longitudinal plasmon

- Density correlation in long-wavelength limit

$$C_{\hat{j}_{p0}^+ \hat{j}_{p0}^+}(\omega, \mathbf{q}) \approx \frac{\chi q^2 (\omega_0^p)^2}{(\omega + i\eta)^2 - (\omega_0^p)^2}$$

$$\chi \equiv - \lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} \frac{1}{q^2} C_{\hat{j}_{p0}^+ \hat{j}_{p0}^+}(\omega, \mathbf{q}) = \frac{\partial P}{\partial E}$$

Static electrical susceptibility

- Coulomb interaction in long-wavelength limit

- Intra-layer $\tilde{V}(\mathbf{q}) = 2\pi/\epsilon q$

- Inter-layer $\tilde{U}(\mathbf{q}) = \tilde{V}(\mathbf{q})e^{-qd}$ $\xrightarrow{qd \rightarrow 0} \tilde{U}(\mathbf{q}) \approx \tilde{V}(\mathbf{q})$

- Effective dielectric function

$$\epsilon_{\text{eff}}(\omega, \mathbf{q}) = \frac{\tilde{V}(\mathbf{q})}{\tilde{V}_{\text{eff}}(\omega, \mathbf{q})} = [1 + \tilde{V}(\mathbf{q}) C_{\hat{j}_{p0}^+ \hat{j}_{p0}^+}(\omega, \mathbf{q})]^{-1}$$

$$q \rightarrow 0 \quad \tilde{V}(\mathbf{q}) C_{\hat{j}_{p0}^+ \hat{j}_{p0}^+}(\omega, \mathbf{q}) \sim q$$

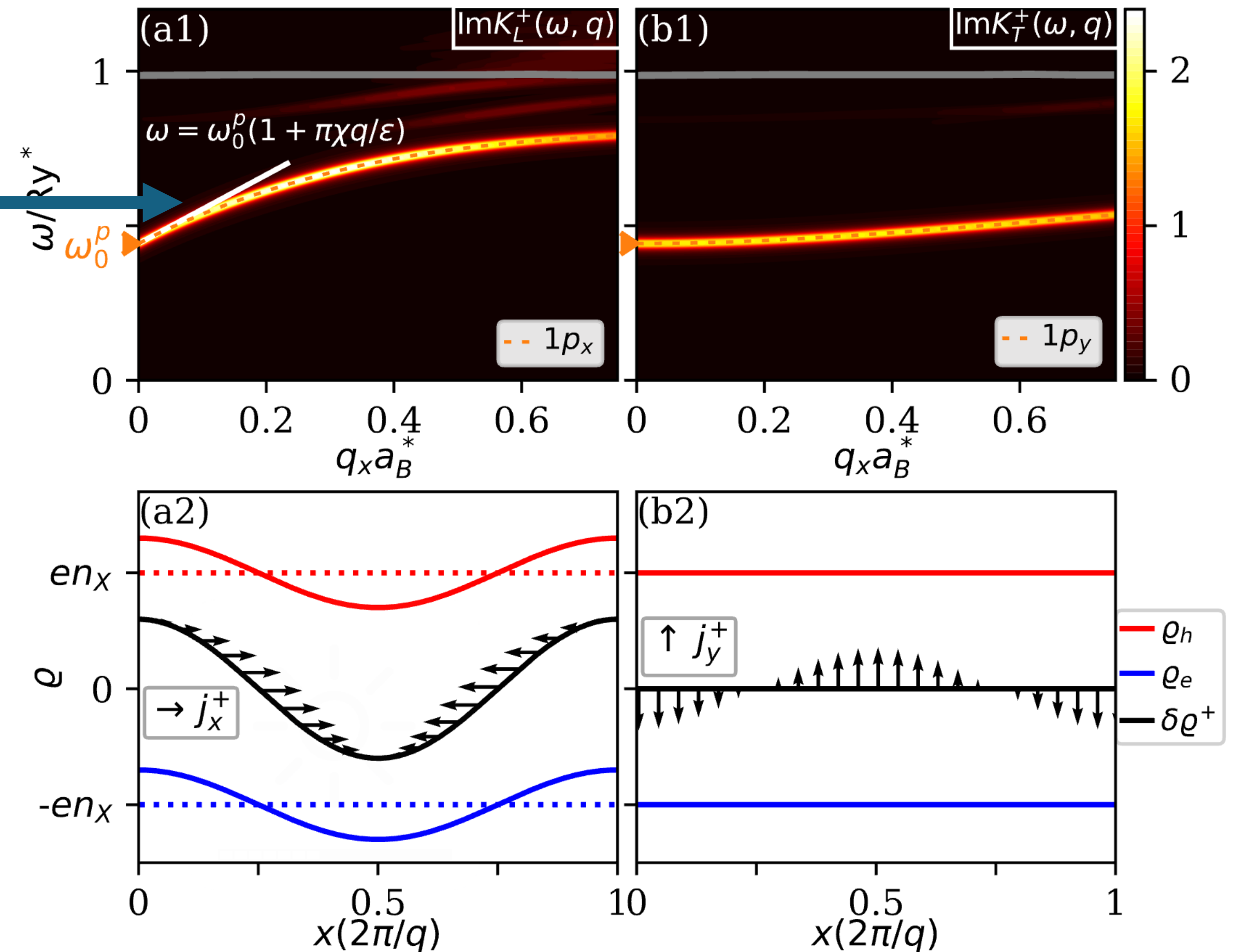
$$\epsilon_{\text{eff}}(\omega, \mathbf{q}) \approx 1 - \tilde{V}(\mathbf{q}) C_{\hat{j}_{p0}^+ \hat{j}_{p0}^+}(\omega, \mathbf{q}) + \mathcal{O}(q^2)$$

- Plasmon dispersion

$$\epsilon_{\text{eff}}(\omega, \mathbf{q}) = 0 \implies \omega \approx \omega_0^p \sqrt{1 + 2\pi\chi q/\epsilon} \approx \omega_0^p (1 + \pi\chi q/\epsilon)$$

TM mode

TE mode



(a1)(a2) Charge density and longitudinal current fluctuations

(b1)(b2) Transverse current fluctuations

2 Response to layer anti-symmetric gauge field

Goldstone mode velocity

- Exciton density correlation function

$$K_{00}^-(\omega, q) = -C_{\hat{j}_{p0}^-, \hat{j}_{p0}^-}(\omega, q) \approx \frac{-\kappa v^2 q^2}{(\omega + i\eta)^2 - v^2 q^2}$$

$$\kappa = \lim_{q \rightarrow 0} \lim_{\omega \rightarrow 0} K_{00}^-(\omega, q) = e^2 \left(\frac{\partial n_X}{\partial \mu_X} \right)_T$$

Static exciton compressibility

- Spatial-components of the response function

$$K_L^-(\omega, q) \approx \frac{\omega^2}{q^2} K_{00}^-(\omega, q) \approx \frac{-\kappa v^2 \omega^2}{(\omega + i\eta)^2 - v^2 q^2}$$

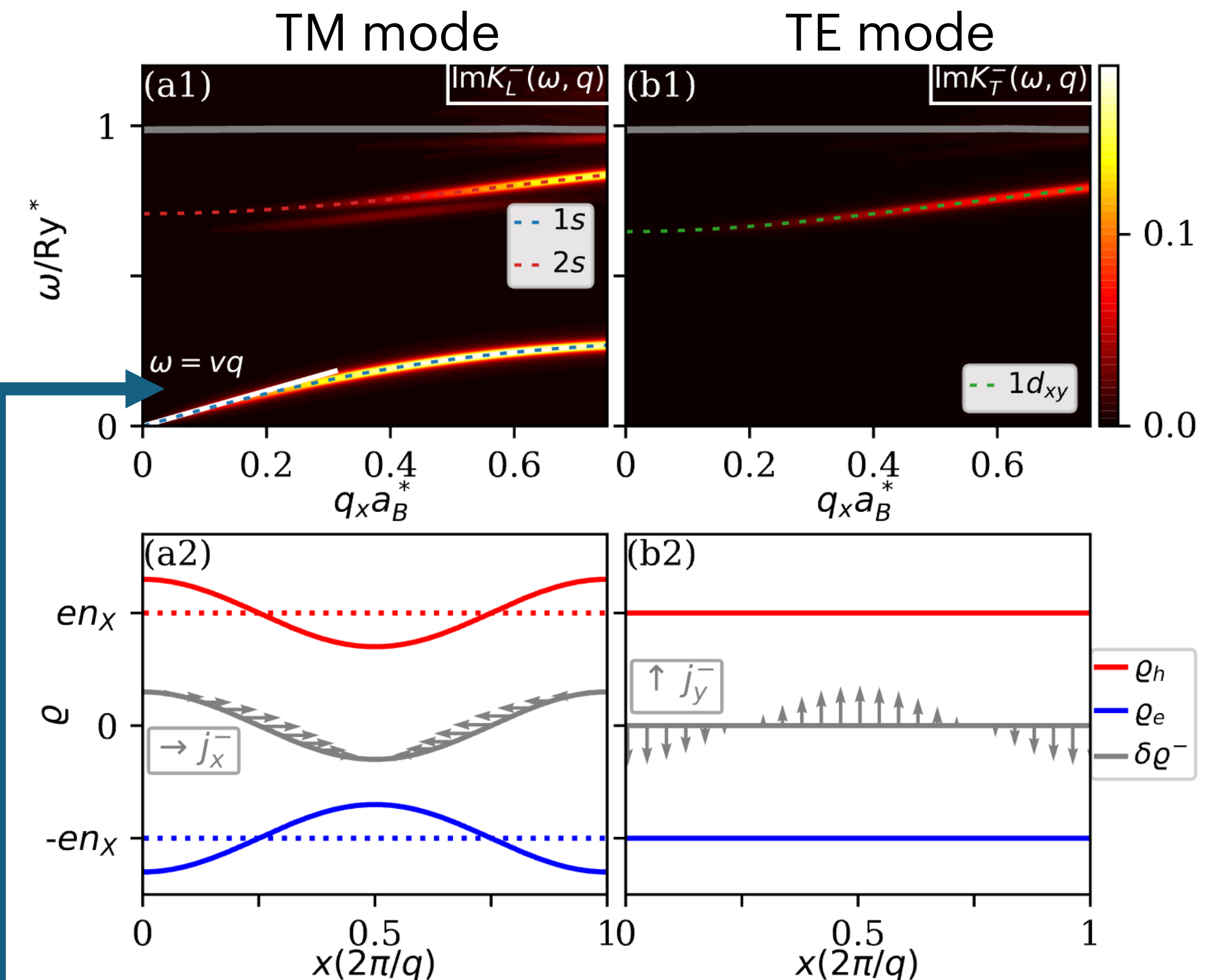
$$K_T^-(\omega, q) = -\frac{e^2 n_X}{4m} + \mathcal{O}(q^2).$$

- Rotation symmetry

$$\lim_{q \rightarrow 0} K_L^-(\omega, q) = \lim_{q \rightarrow 0} K_T^-(\omega, q)$$

- Goldstone mode velocity

$$\kappa v^2 = \frac{e^2 n_X}{4m} \implies v = \sqrt{\frac{e^2 n_X}{4m\kappa}}$$



(a1)(a2) Exciton density and longitudinal current fluctuations

(b1)(b2) Transverse current fluctuations

Detecting the Goldstone mode (Exciton Meissner effect) via microwave impedance microscopy (MIM)

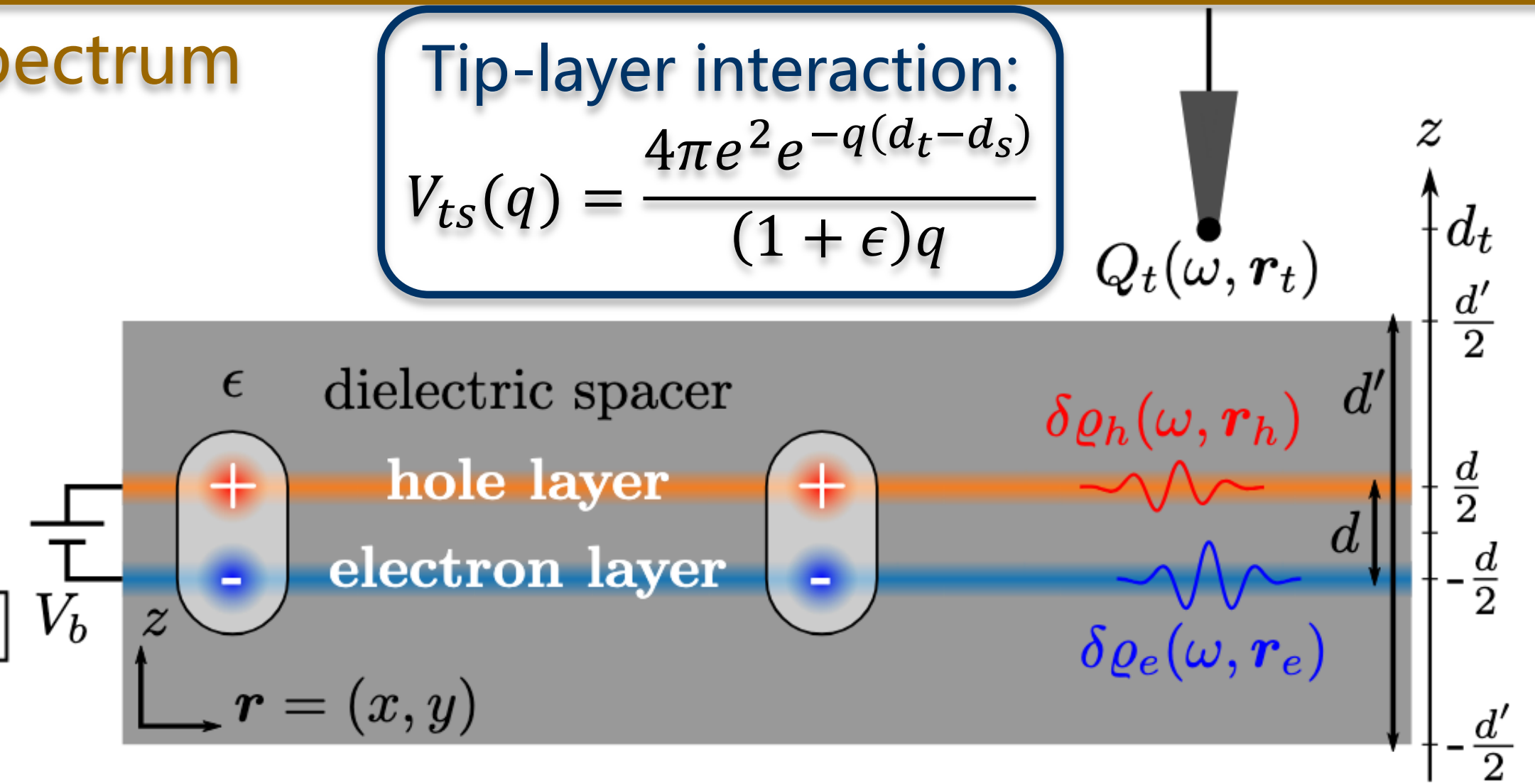
MIM spectrum

- Quantum correction to **real** admittance

$$U_t(\omega) = Q_t(\omega)/C_t + \delta U(\omega)$$

$$Y_t(\omega) \equiv I_t(\omega)/U_t(\omega) = -i\omega Q_t(\omega)/U_t(\omega)$$

$$\text{Re}[\delta Y_t(\omega)] = \omega C_t^2 \mathcal{V}^{-1} \sum_{\sigma=\pm, \mathbf{q}} [\tilde{V}^\sigma(\mathbf{q})]^{-2} \text{Im}[K_{00}^\sigma(\omega, \mathbf{q})]$$



- Low frequency ($\omega < \omega_0^p \sim 10\text{meV}$ for bilayer EI), only has Goldstone mode contribution

$$K_{00}^-(\omega, \mathbf{q}) \approx \frac{\kappa v^2 q^2}{v^2 q^2 - \omega^2} + \frac{i\pi \kappa v q}{2} [\delta(\omega - vq) - \delta(\omega + vq)]$$

$$\tilde{V}^-(\mathbf{q}) = \frac{4\pi e^{-qd_t} (e^{-qd/2} - e^{qd/2})}{(1+\epsilon)q} \approx -\frac{4\pi d e^{-qd_t}}{(1+\epsilon)}$$

$$\text{Re}[\delta Y(\omega)] \sim \omega^3 e^{-\frac{2d_t\omega}{v}}$$

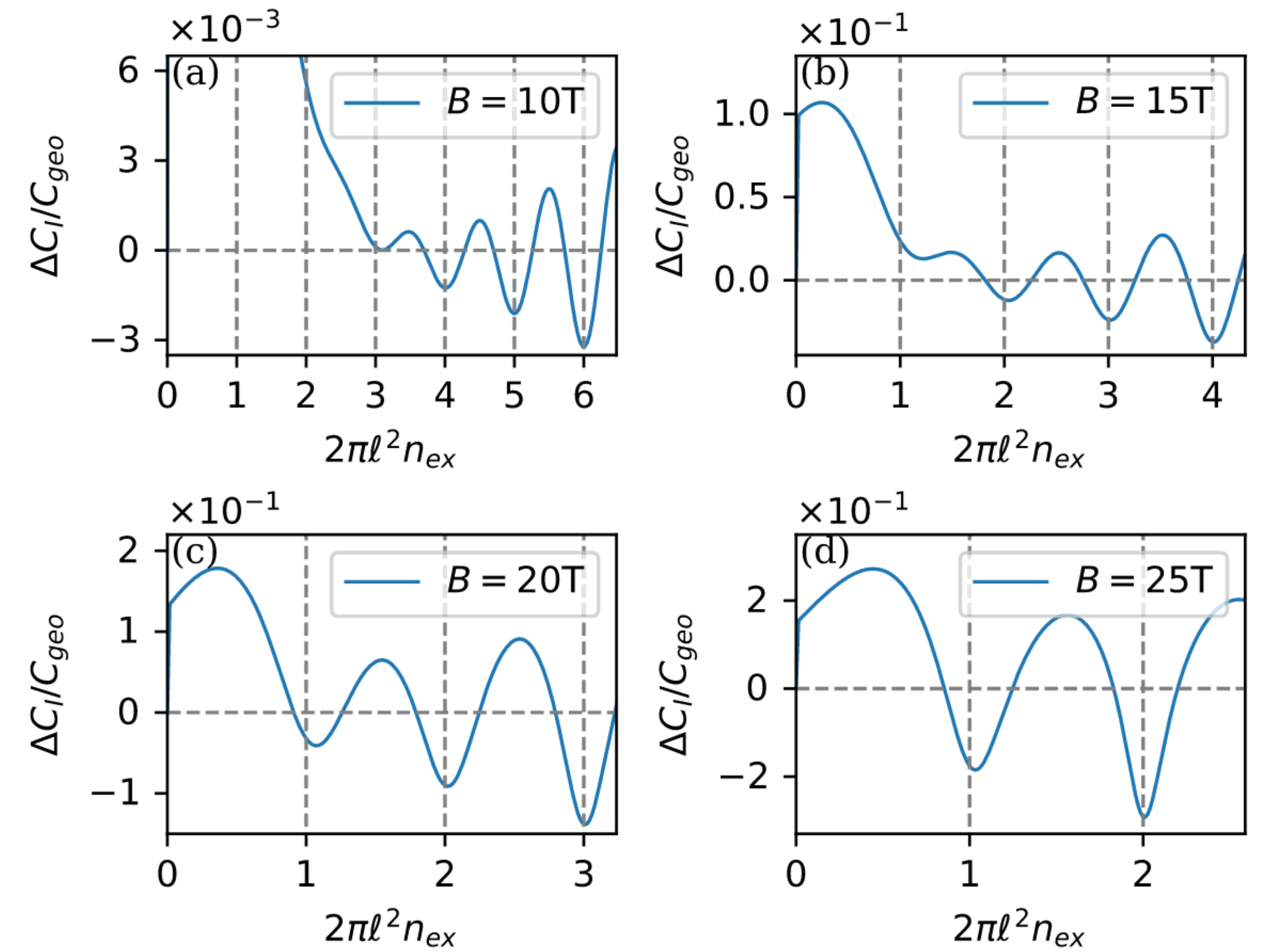
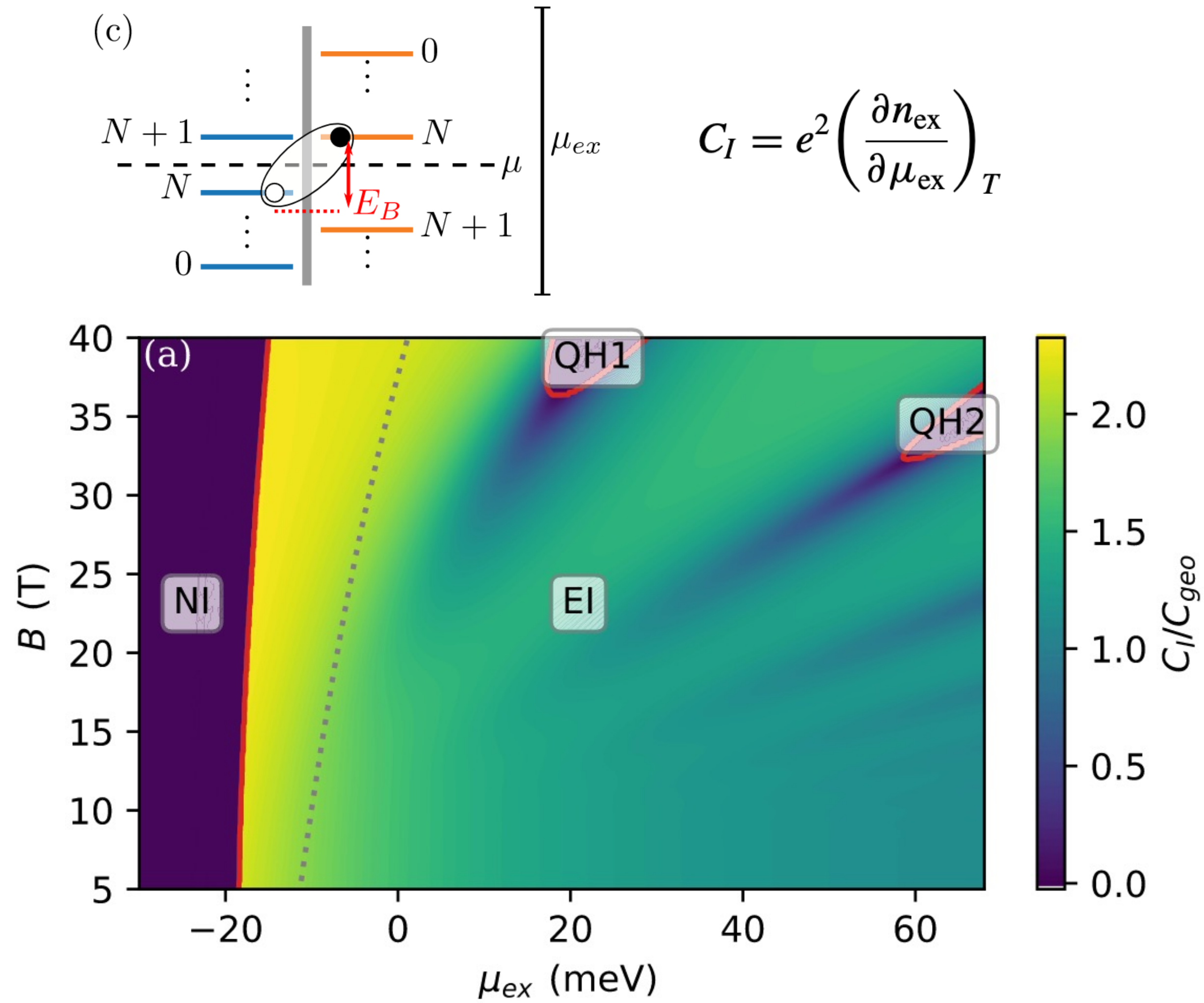
- Nonzero: Gapless excitations
- ω^3 dependence: Linear dispersion
- Exponential factor: Determine velocity

$$\text{Re}[\delta Y_t(\omega)] = \frac{\omega C_t^2 (4\pi d)^2 \pi \kappa v}{2(1+\epsilon)^2} \int_0^\infty \frac{qdq}{2\pi} e^{-2qd_t} q \delta(\omega - vq) = \frac{4\pi^2 d^2 C_t^2 \kappa}{(1+\epsilon)^2 v^2} \omega^3 e^{-2d_t\omega/v}$$



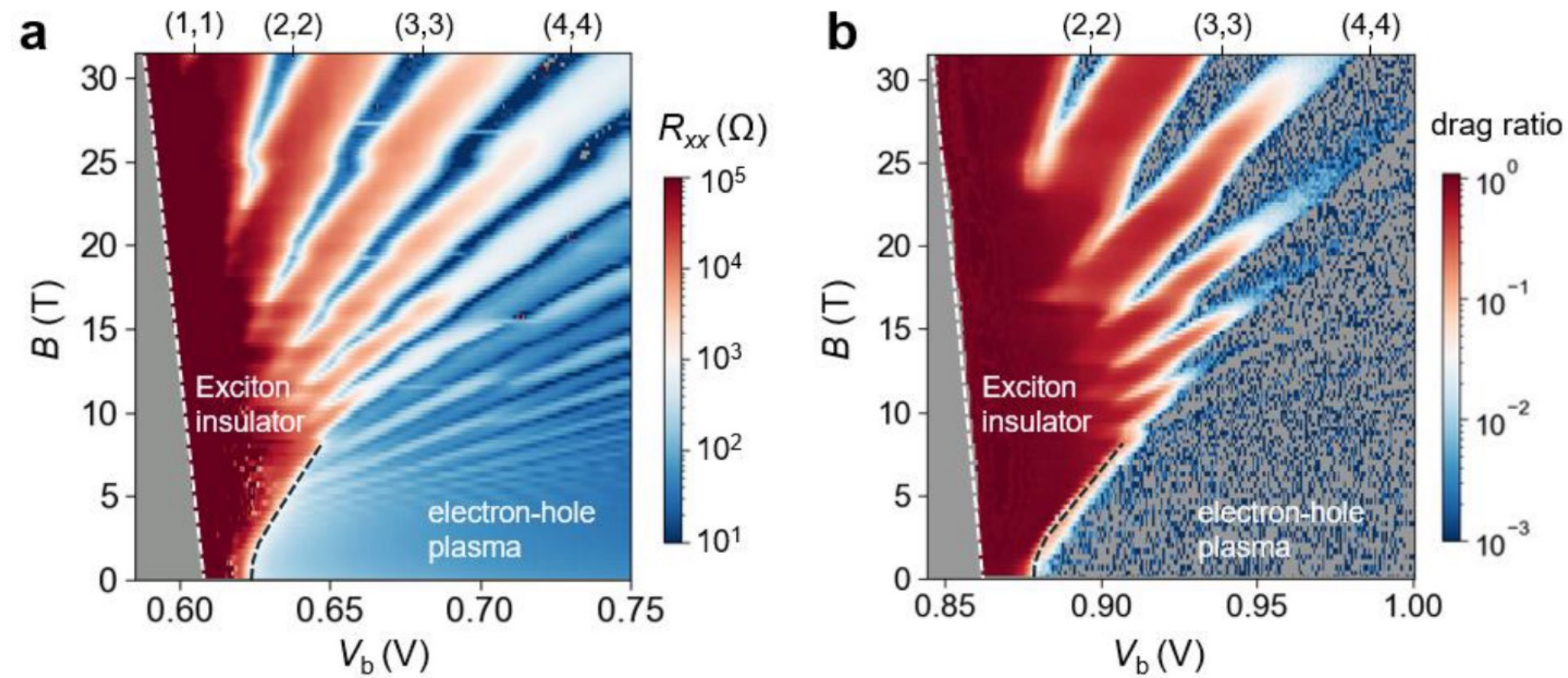
Bilayer EI With Perpendicular Magnetic field

Mean-field phase diagram and quantum oscillation

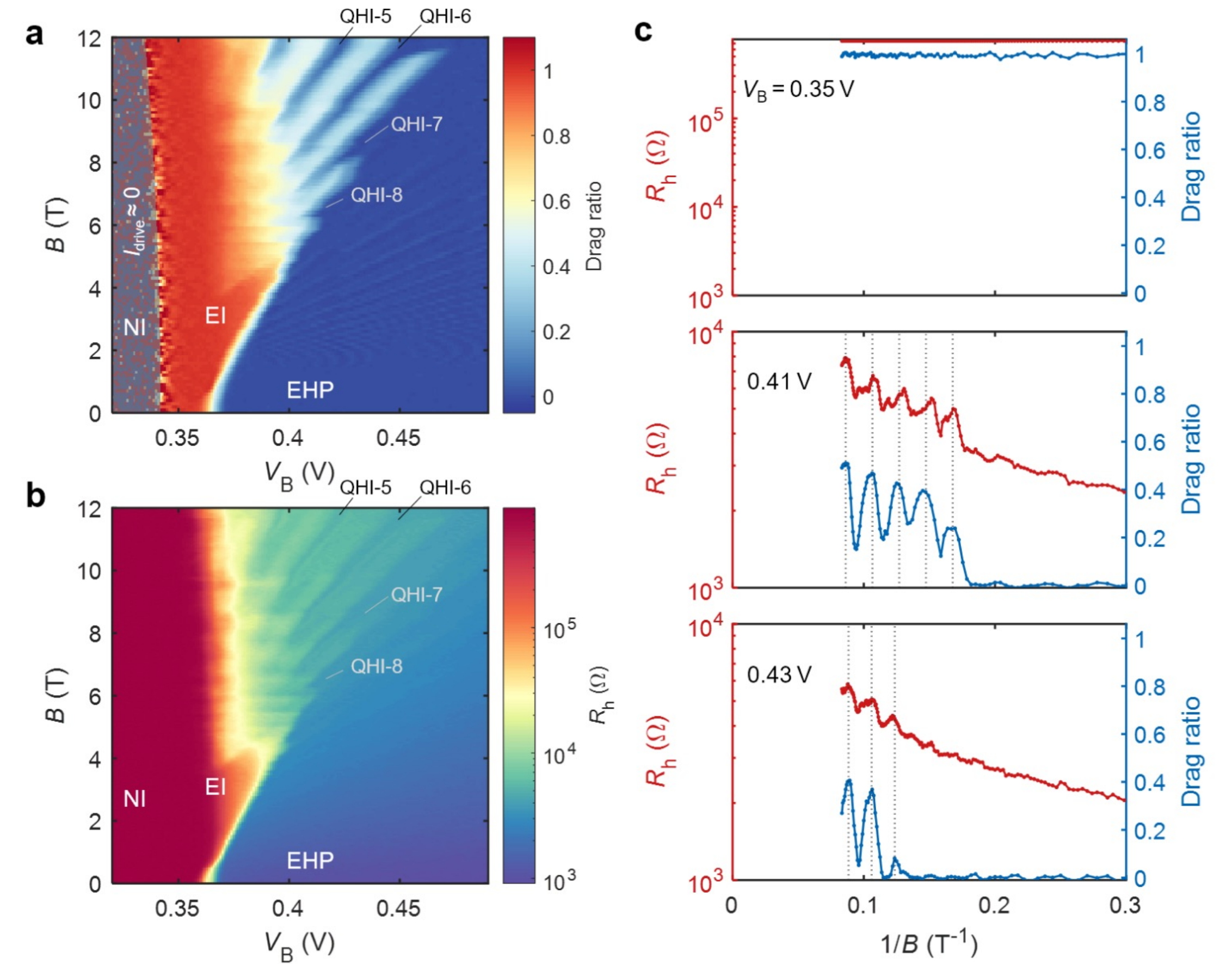


Y. Shao and X. Dai, Quantum oscillations in an excitonic insulating electron-hole bilayer, Phys. Rev. B **109**, 155107 (2024).

Mean-field phase diagram and quantum oscillation

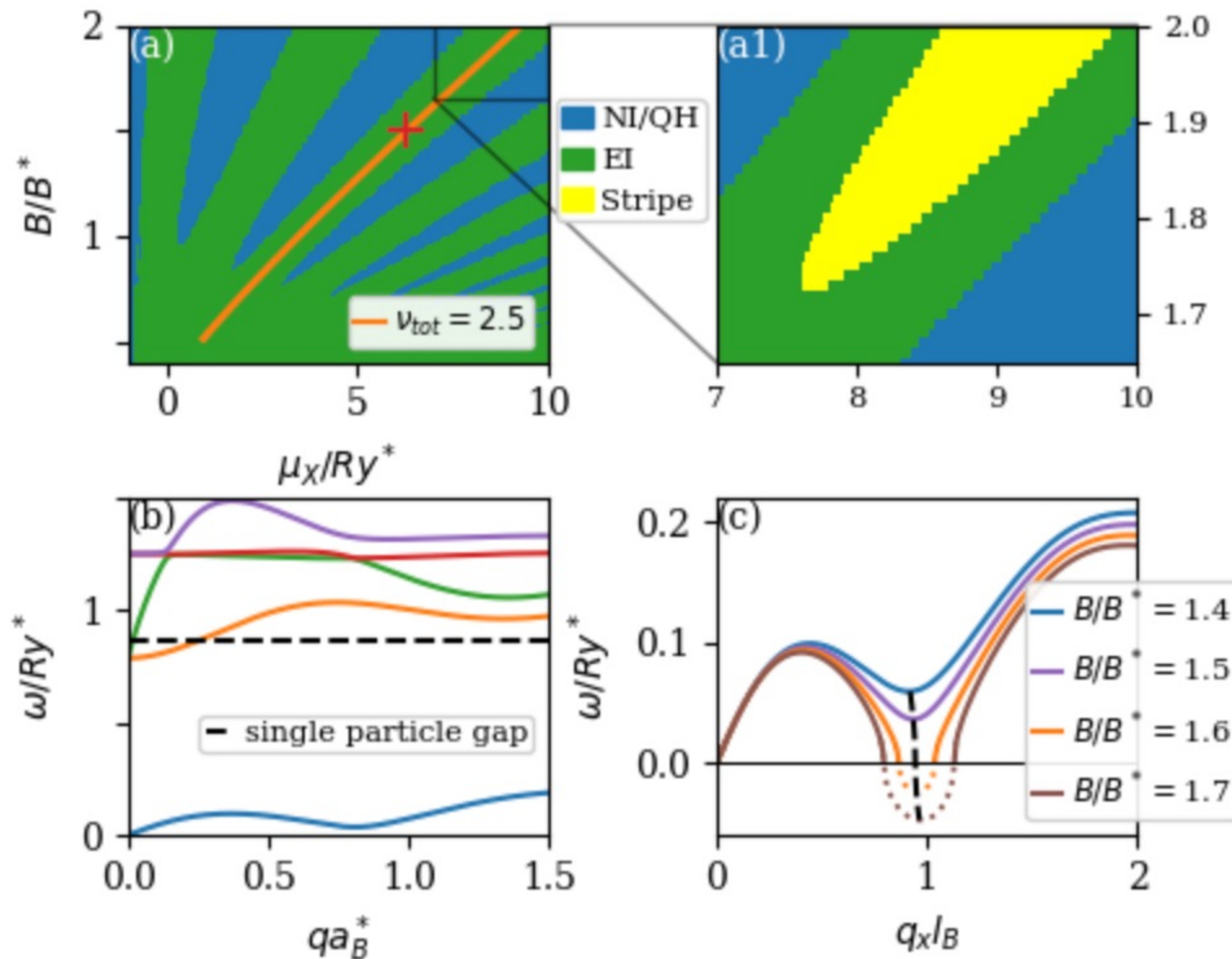


P. X. Nguyen, R. Chaturvedi, B. Zou, K. Watanabe, T. Taniguchi, A. H. MacDonald, K. F. Mak, and J. Shan, *Quantum Oscillations in a Dipolar Excitonic Insulator*, arXiv:2501.17829.



R. Qi et al., *Competition between Excitonic Insulators and Quantum Hall States in Correlated Electron-Hole Bilayers*, arXiv:2501.18168.

Phase diagram and collective modes with perpendicular magnetic field

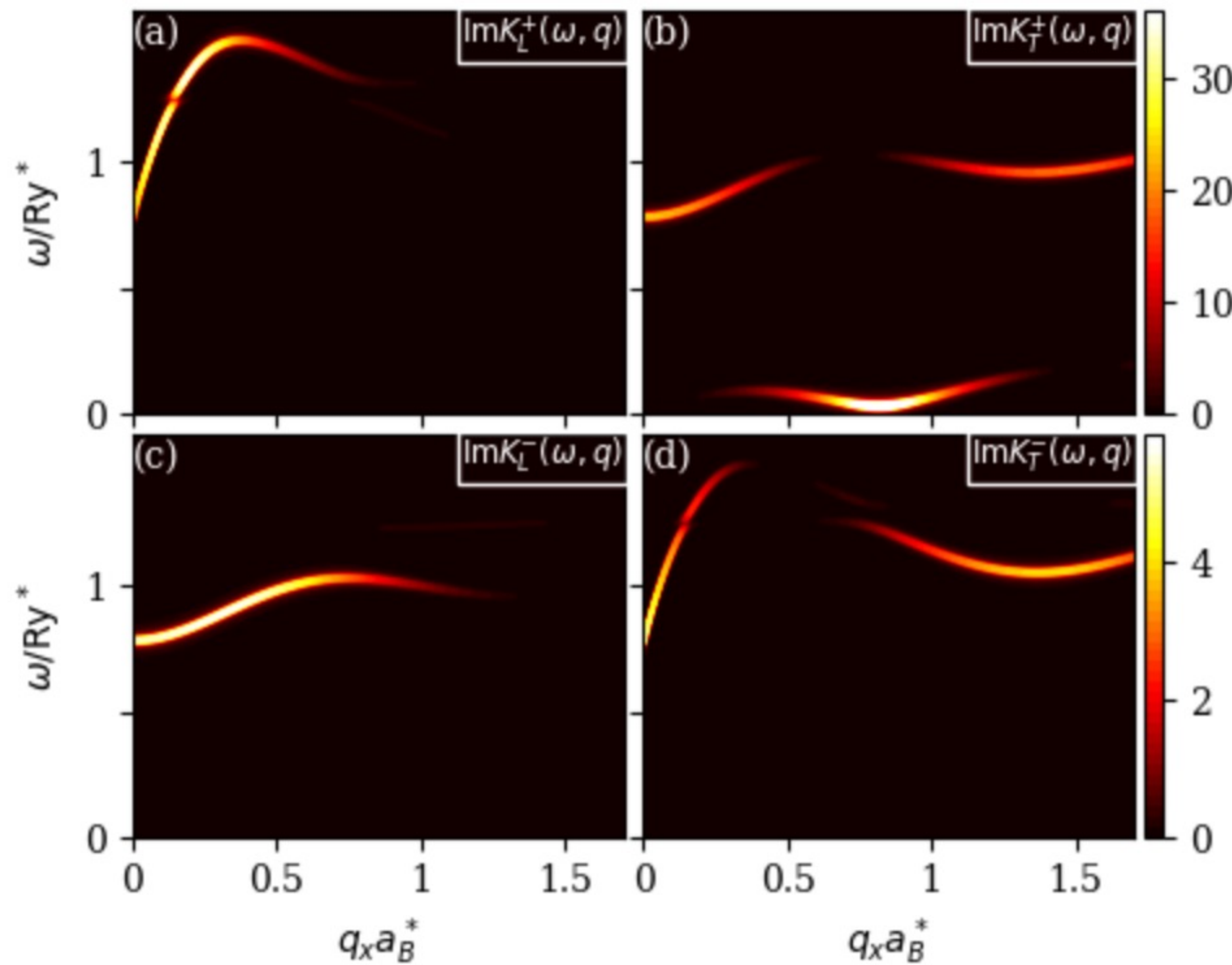


- (a) Phase diagram (only uniform state considered)
- (b) Collective mode spectrum at $\nu_{tot} = 2.5, \frac{B}{B^*} = 1.5$ (red plus symbol in (a))
- (c) Goldstone mode dispersion along the orange line in (a)
- (a1) Phase diagram with nonuniform state

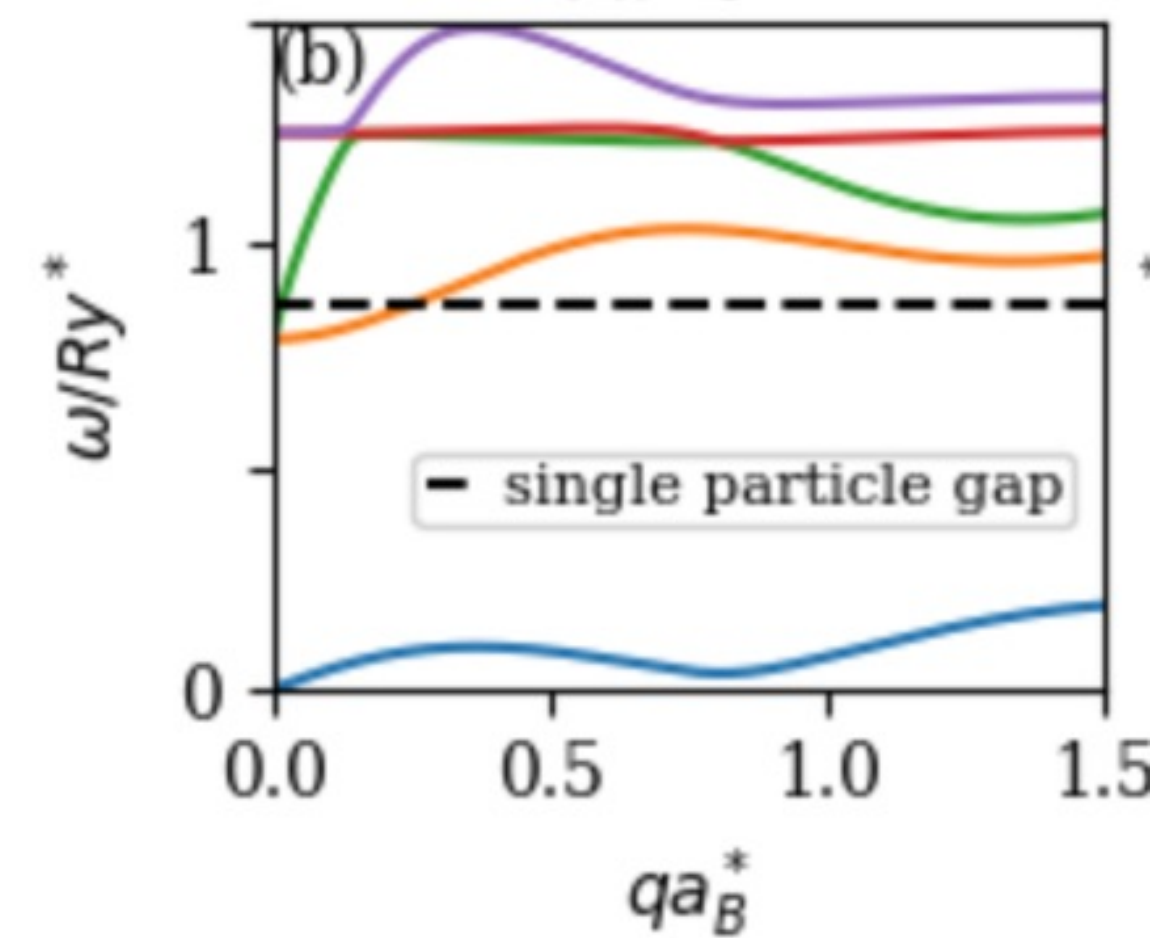
- **Magnetic roton**
- **The minimum $Q_c l_B$ mainly determined by the Landau level form factor**

$$\begin{aligned} Ry^* &\approx 30 \text{ meV} \\ a_B^* &\approx 2.4 \text{ nm} \\ B^* &\approx 55 \text{ T} \end{aligned}$$

Symmetric part of the response function

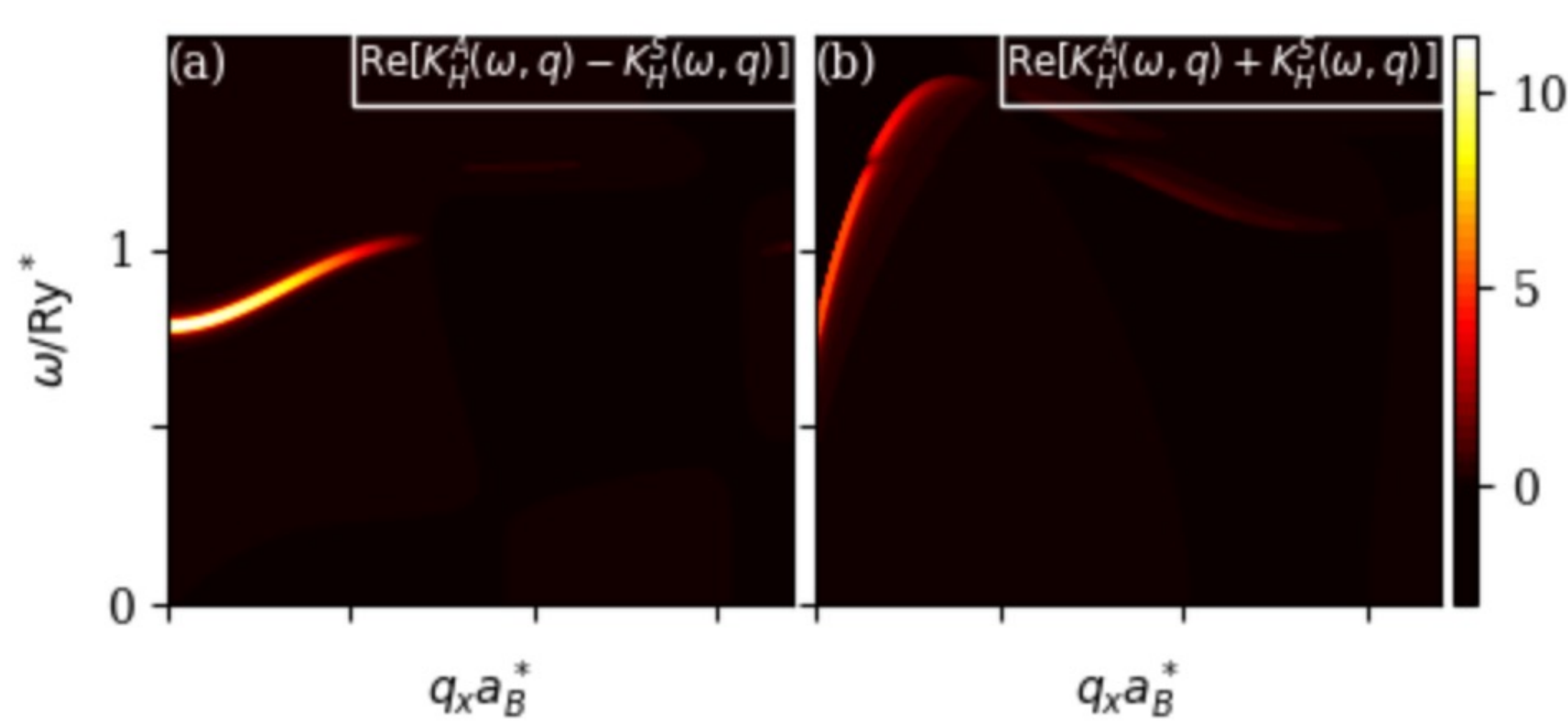


$$K_{\mu\nu}^{\sigma\sigma} = \begin{bmatrix} \frac{q^2}{\omega^2} K_L^\sigma & -\frac{q}{\omega} K_L^\sigma & 0 \\ -\frac{q}{\omega} K_L^\sigma & K_L^\sigma & 0 \\ 0 & 0 & K_T^\sigma \end{bmatrix}$$



- Different modes strongly hybridized at finite q
- Roton mainly contributed by the transverse plasmon

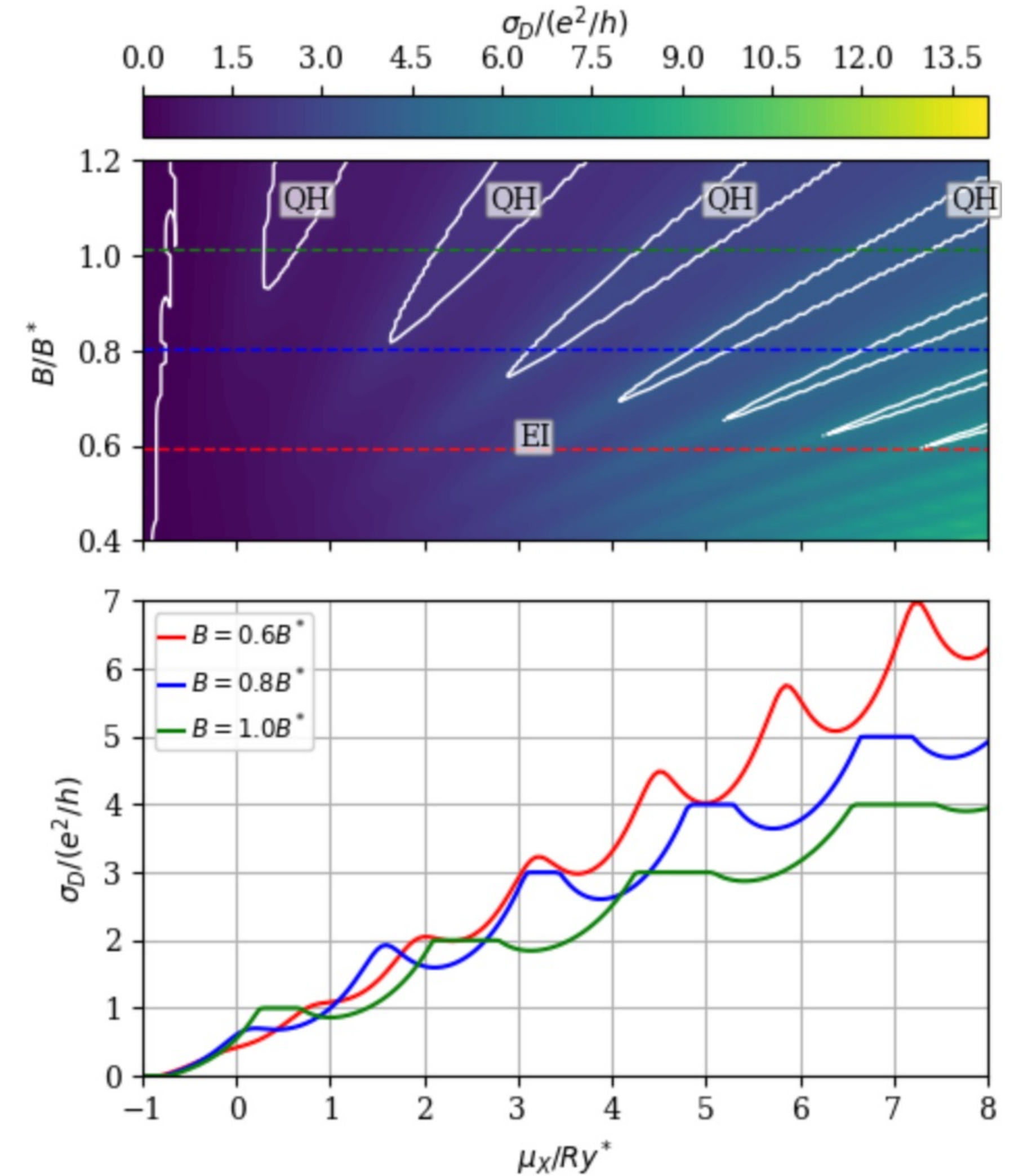
Anti-symmetric part of the response function and dipole Hall effect



$$K_{\mu\nu}^{+-} = \begin{bmatrix} 0 & 0 & -\frac{q}{\omega}(K_H^A + K_H^S) \\ 0 & 0 & K_H^A + K_H^S \\ \frac{q}{\omega}(K_H^A - K_H^S) & -K_H^A + K_H^S & 0 \end{bmatrix}$$

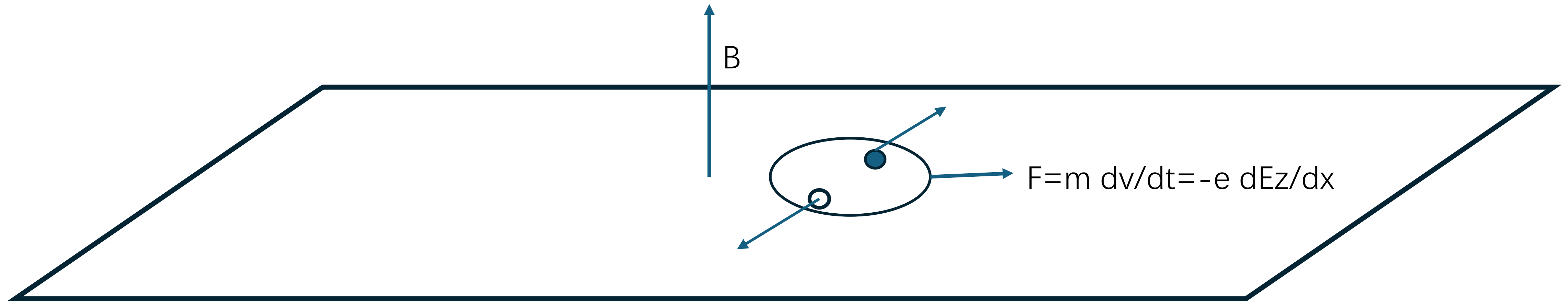
$$K_{\mu\nu}^{-+} = -K_{\nu\mu}^{+-}$$

- $j^- = K^{-+}A^+$: dipole Hall effect
- $j^+ = K^{+-}A^-$: inverse dipole Hall effect



$$\sigma_D = \lim_{\omega \rightarrow 0} \frac{1}{i\omega} \lim_{q \rightarrow 0} K_H^A(\omega, q)$$

Classical picture of inverse Dipole Hall effect



The C.O.M motion v generates a net Lorentz force acting on the relative motion between the e and h , which causes a electric polarization P through the “Stark effect” $P=\chi_e B v$

So the transverse current $I=\chi_e e^2 B/m \, dE_z/dx$

Consider an exciton with center of mass velocity $\mathbf{v}_X$. It will be accelerated under the driving force for the inter-layer excitons, $\mathbf{F}_X = -e\vec{\nabla}_{2D}E_z$, leading to the following equation of motion with a damping term,

$$m_X\dot{\mathbf{v}}_X = -m_X\gamma\mathbf{v}_X - e\vec{\nabla}_{2D}E_z \quad (1)$$

Assume the time periodical driving force with frequency ω , we get

$$\begin{aligned} -i\omega m_X\mathbf{v}_X &= -m_X\gamma\mathbf{v}_X - e\vec{\nabla}_{2D}E_z \\ \mathbf{v}_X &= \frac{-e\vec{\nabla}_{2D}E_z}{(\gamma - i\omega)m_X} \Rightarrow \mathbf{v}_X = \frac{e\mathbf{E}_{2D}^{(-)}}{(\gamma - i\omega)m_X} \end{aligned} \quad (2)$$

When an additional magnetic field is applied along the c-axis, the electron and hole which are bound into the exciton will receive the Lorentz force pointing to opposite directions due to the opposite charges they are carrying, namely $\mathbf{F}_e = -\mathbf{F}_h = e\mathbf{v}_X \times \mathbf{B}$. Then such a Lorentz force will simply act on the relative motion of the electron-hole pair and lead to electric polarization as

$$\mathbf{P}_X = en_X(\langle \mathbf{r}_h \rangle - \langle \mathbf{r}_e \rangle) = \chi_e e\mathbf{v}_X \times \mathbf{B} = \frac{e^2 n_X \chi_e \mathbf{E}_{2D}^{(-)} \times \mathbf{B}}{(\gamma - i\omega)m_X} \quad (3)$$

Finally the current contributed by all the exciton can be expressed as

$$\mathbf{J}_e = \dot{\mathbf{P}}_X = -i\omega\mathbf{P}_X = \frac{-i\omega}{(\gamma - i\omega)m_X} e^2 \chi_e n_X \mathbf{E}_{2D}^{(-)} \times \mathbf{B} \quad (4)$$

In the superfluid phase $\gamma = 0$ and we get

$$\mathbf{J}_e = \frac{e^2 \chi_e n_X}{m_X} \mathbf{E}_{2D}^{(-)} \times \mathbf{B} = \omega_{XP}^2 \chi_e \mathbf{E}_{2D}^{(-)} \times \mathbf{B} \quad (5)$$

Static limit: Meissner effect under Bz field manifested as a special magnetic electric effect

In the strict static case we get

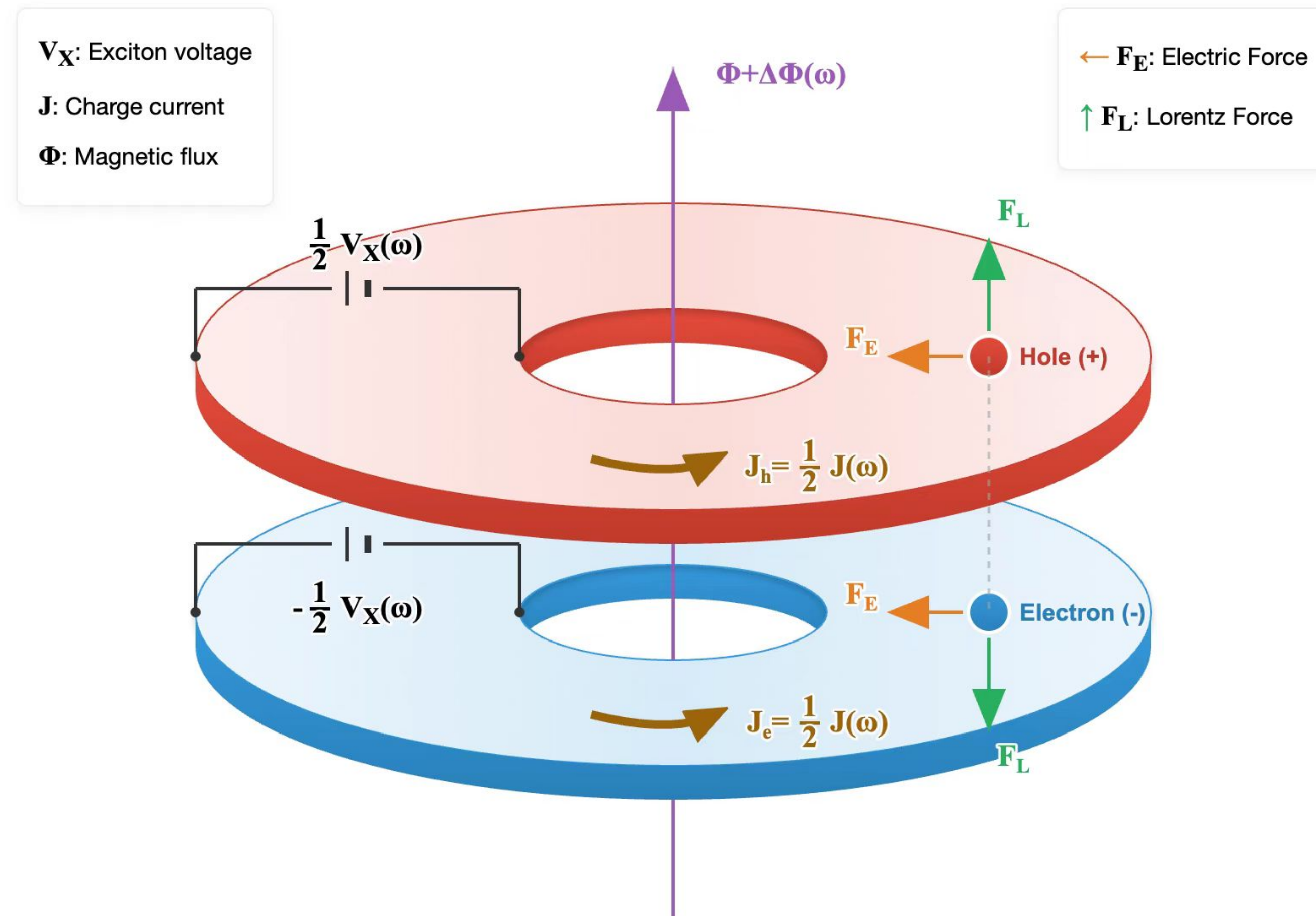
$$\mathbf{J}_e = \frac{e^2 \chi_e n_X}{m_X} \mathbf{E}_{2D}^{(-)} \times \mathbf{B} = \omega_{XP}^2 \chi_e \mathbf{E}_{2D}^{(-)} \times \mathbf{B} \Rightarrow \quad (5)$$
$$\frac{d}{dt}(\mathbf{P}_X - \omega_{XP}^2 \chi_e \mathbf{A}_X \times \hat{e}_z B_z) = 0 \Rightarrow \mathbf{P}_X - \omega_{XP}^2 \chi_e \mathbf{A}_X \times \hat{e}_z B_z = \text{Constant}$$

While from the exciton Meissner effect or exciton second London equation this Constant is ZERO!

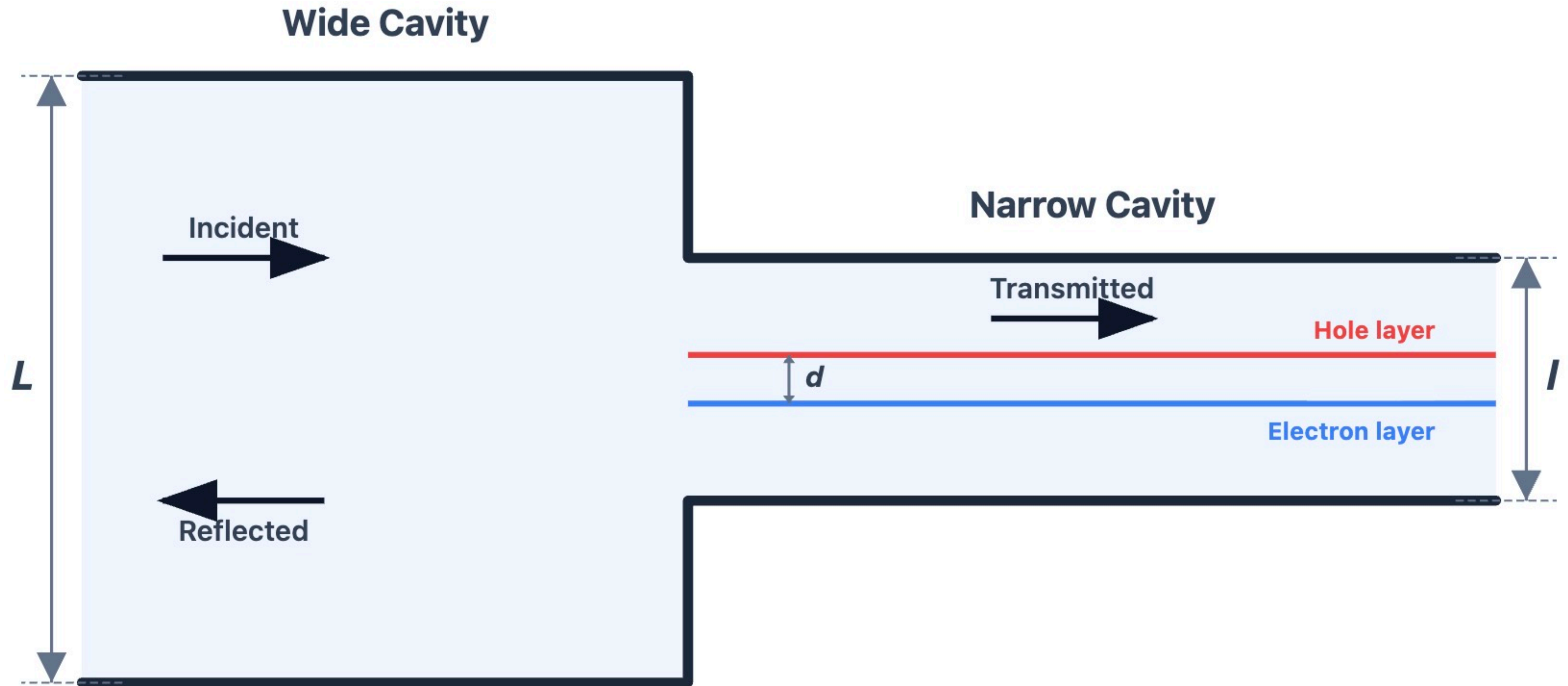
So we have

$$\mathbf{P}_X = \omega_{XP}^2 \chi_e \mathbf{B}_{\parallel} B_z \quad (6)$$

Detect Exciton Meissner effect by Corbino geometry



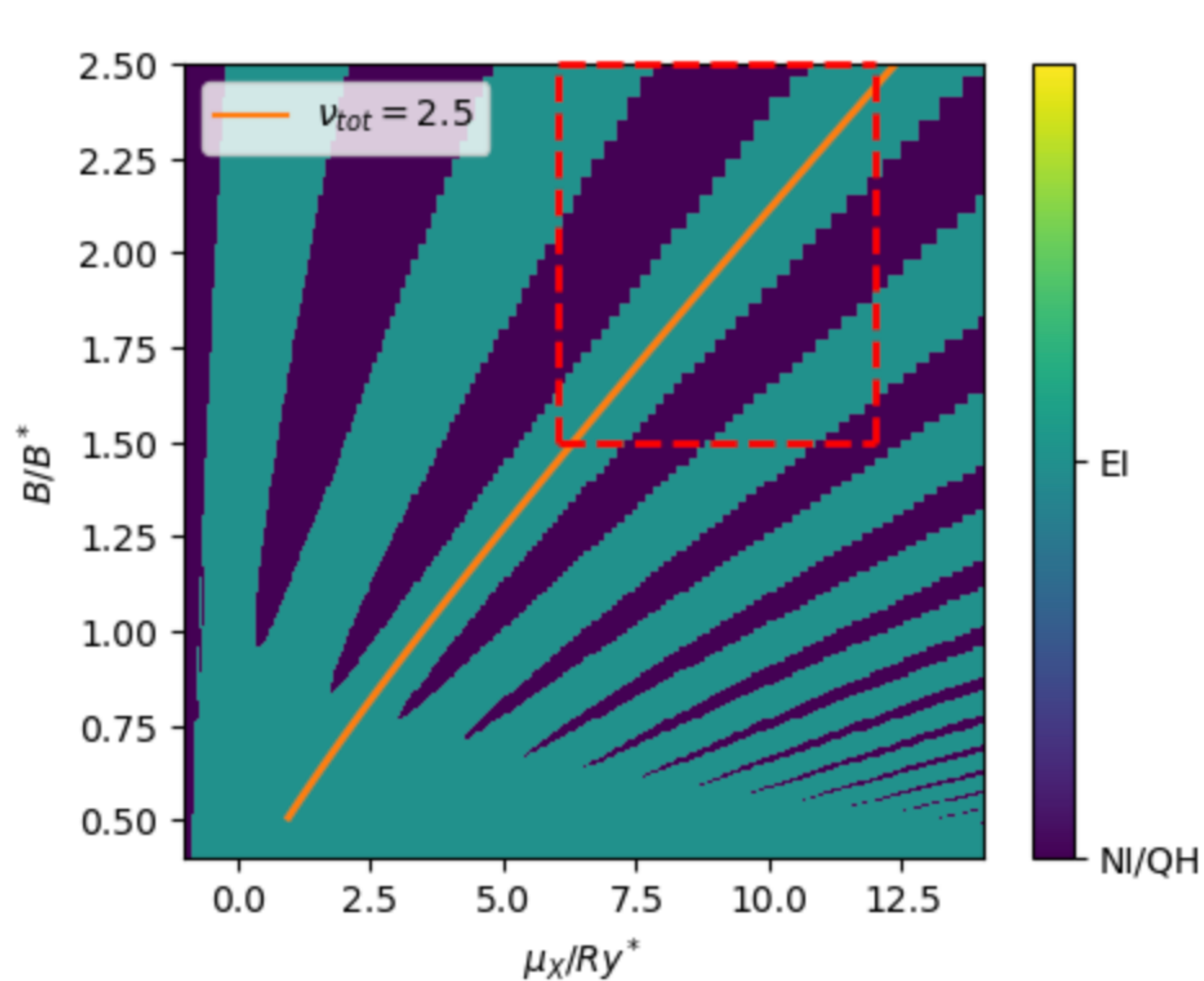
Detect the Goldstone mode by Waveguide



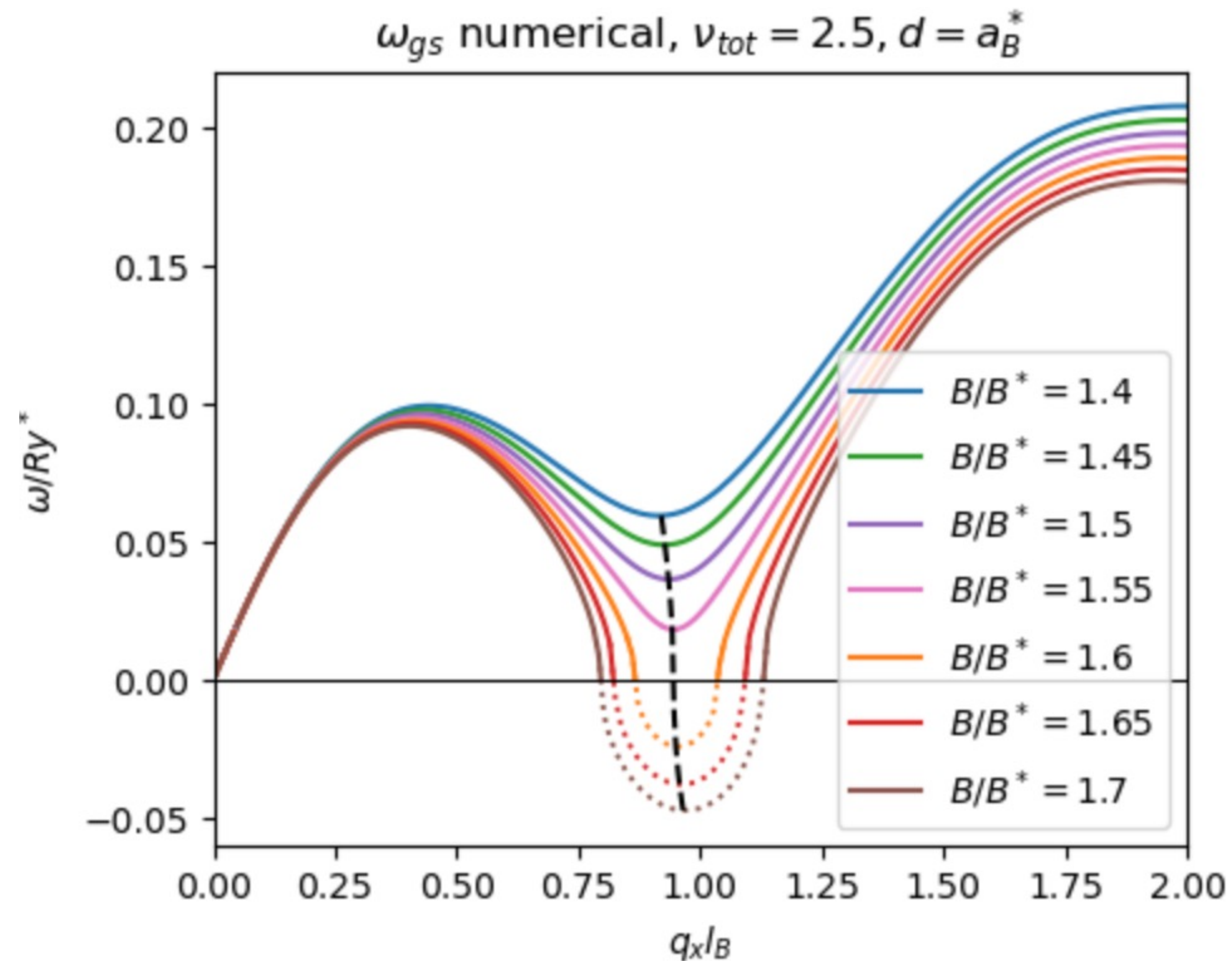
Summary

- There are fruitful EM response for EI (superfluid) phase
- Gapped plasmon modes to be detected by optical means
- Gapless Goldenstone modes to be detected by the near field optics (MIMS)
- The exciton London equation can be demonstrated by inverse hall effect

Softening of the Goldstone mode (roton instability)

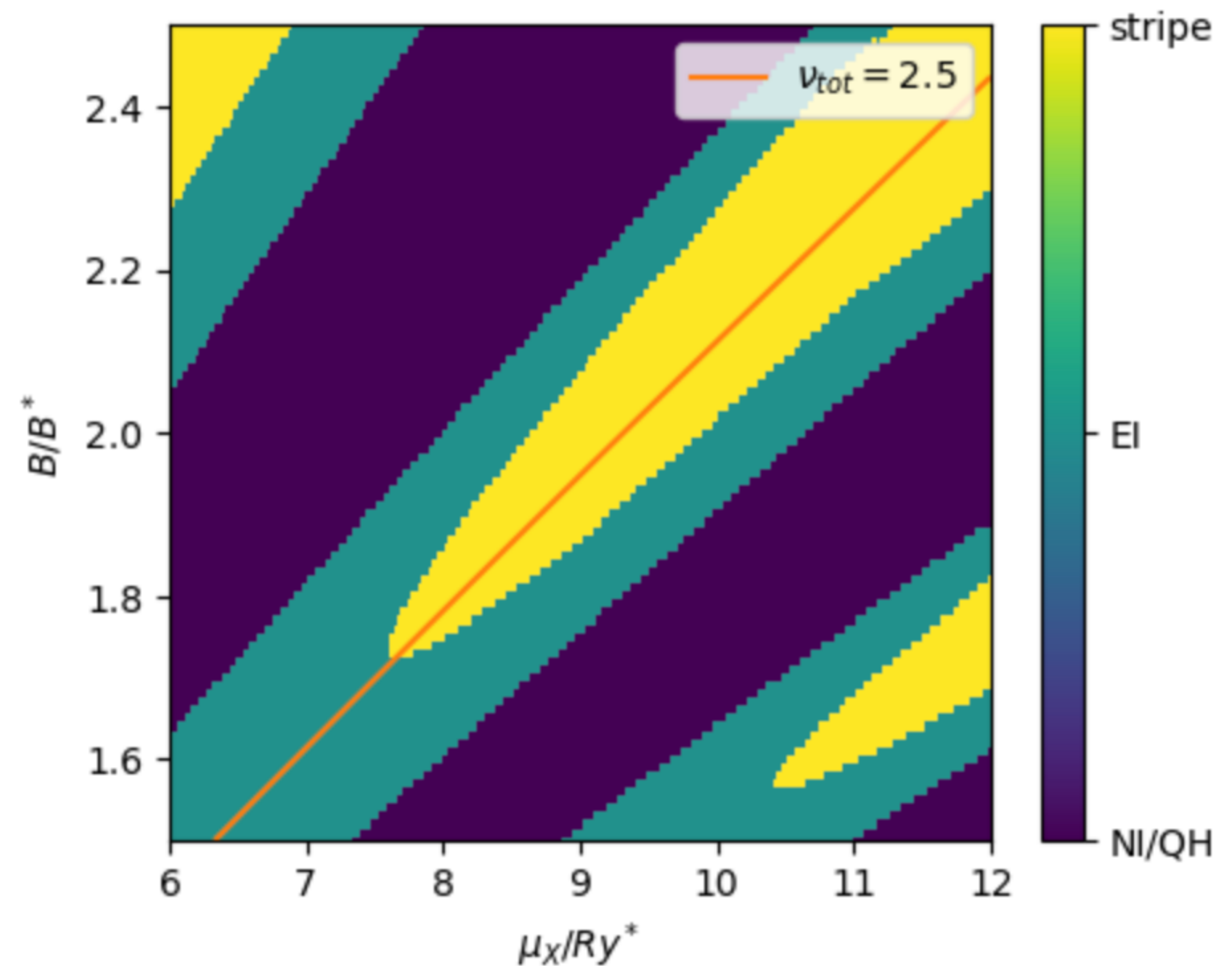
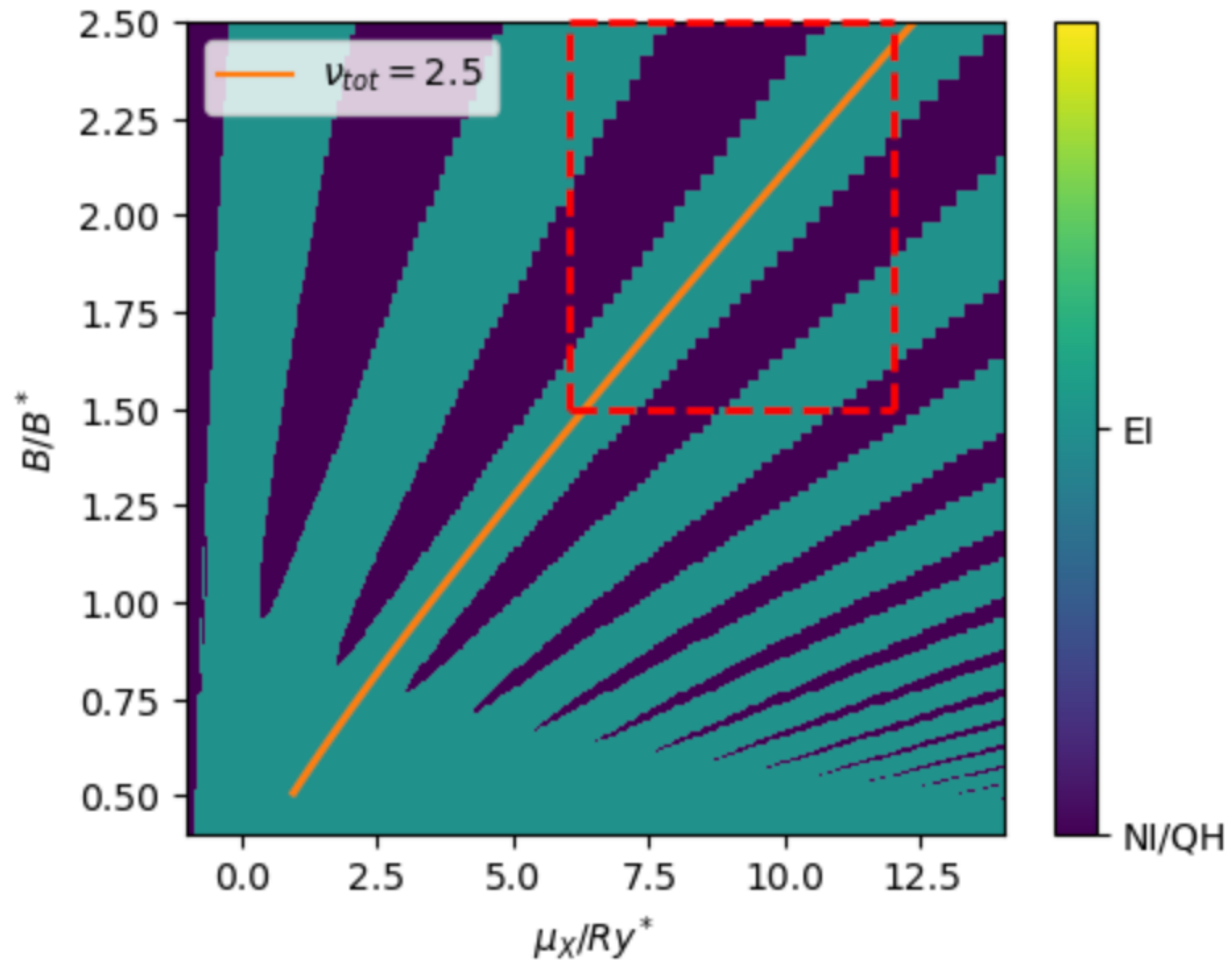


$$\begin{aligned} Ry^* &\approx 30 \text{ meV} \\ a_B^* &\approx 2.4 \text{ nm} \\ B^* &\approx 55 \text{ T} \end{aligned}$$



$Q_c l_B$ mainly determined by the Landau level form factor

Nonuniform EI (super solid state)



$$\begin{aligned} Ry^* &\approx 30\text{meV} \\ a_B^* &\approx 2.4\text{nm} \\ B^* &\approx 55\text{T} \end{aligned}$$

1 Collective modes: time-dependent Hartree Fock

Feynman diagrammatic representation

- HF green function

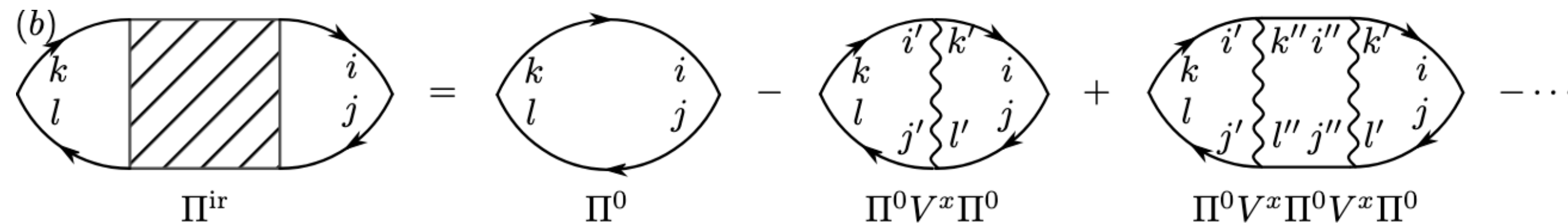
$$\mathcal{G}_{ij}(i\omega_m) = \frac{\delta_{ij}}{i\omega_m - \xi_i}$$

- "Non-interaction" polarization function (bubble diagram)

$$\Pi_{ij,kl}^0(i\nu_n) = \frac{1}{\beta} \sum_{i\omega_m} \mathcal{G}_{ik}(i\omega_m + i\nu_n) \mathcal{G}_{lj}(i\omega_m) = \frac{\delta_{(ij)(kl)} f_{ji}}{i\nu_n - (\xi_i - \xi_j)}$$

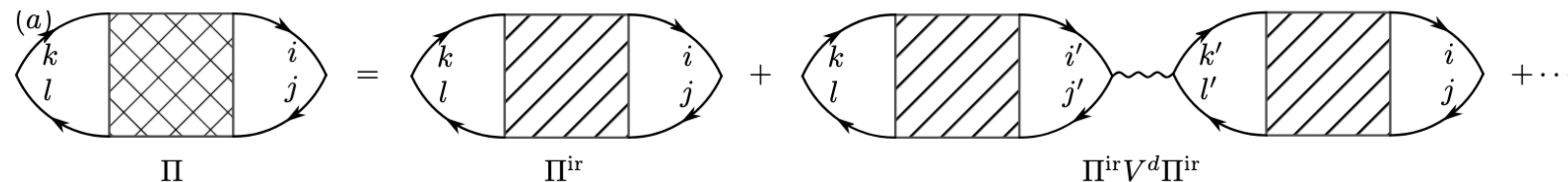
- Irreducible polarization function (vertex correction under ladder approximation)

$$\Pi^{\text{ir}} = \Pi^0 - \Pi^0 V^x \Pi^0 + \Pi^0 V^x \Pi^0 V^x \Pi^0 - \dots = \Pi^0 (1 - V^x \Pi^{\text{ir}}) = (1 - \Pi^{\text{ir}} V^x) \Pi^0$$



- Polarization function

$$\Pi = \Pi^{\text{ir}} + \Pi^{\text{ir}} V^d \Pi^{\text{ir}} + \Pi^{\text{ir}} V^d \Pi^{\text{ir}} V^d \Pi^{\text{ir}} + \dots = \Pi^{\text{ir}} (1 + V^d \Pi) = (1 + \Pi V^d) \Pi^{\text{ir}}$$



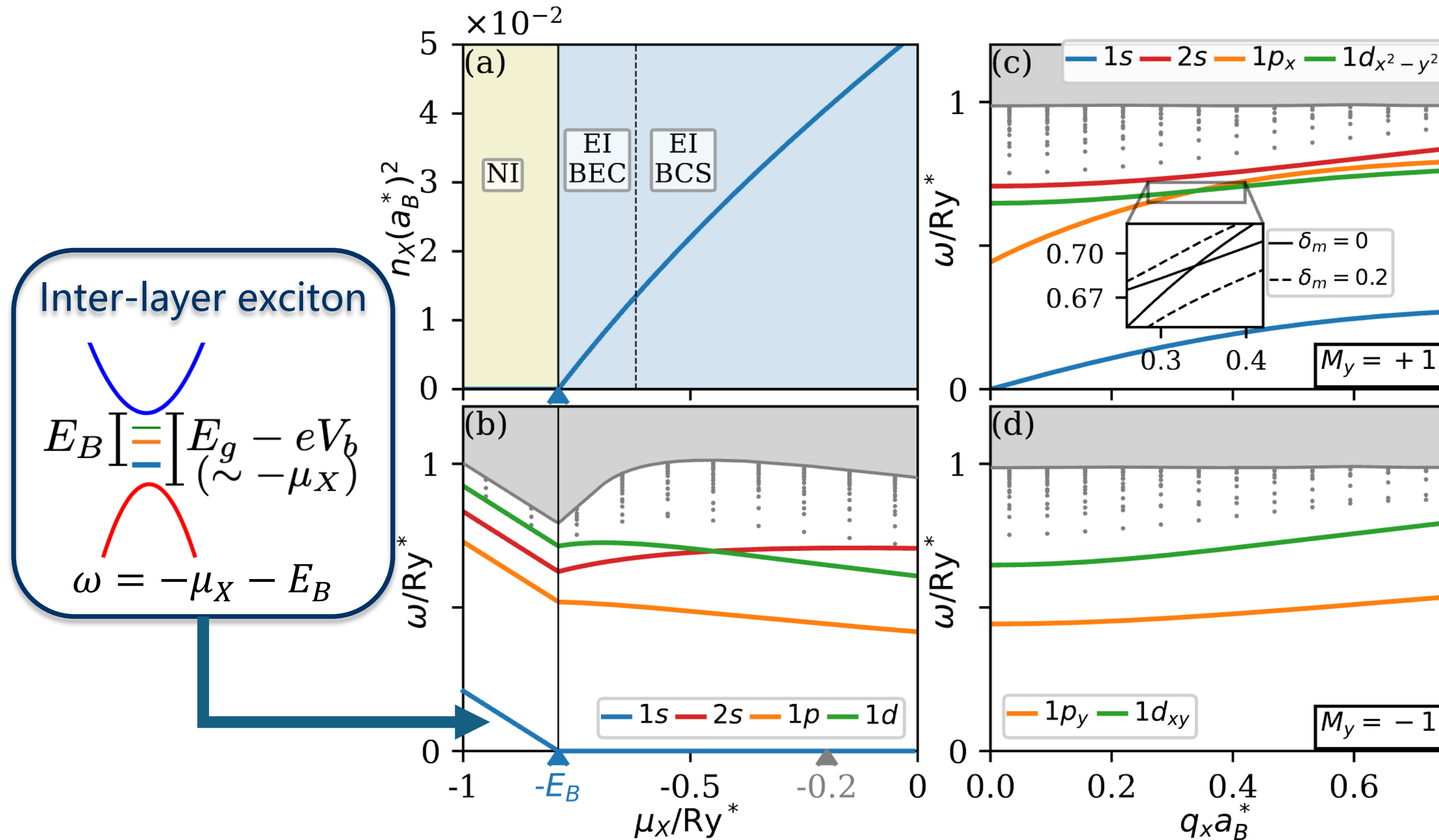
Notations

- ξ_i : quasi-particle energy
- $f_i = n_F(\xi_i)$: occupation number
- $f_{ji} = f_j - f_i$
- $\delta_{(ij)(kl)} = \delta_{ik} \delta_{jl}$

Direct/exchange interaction

$$V_{k'l',i'j'}^d = \langle k', j' | V | l', i' \rangle$$

$$V_{k'l',i'j'}^x = \langle k', j' | V | i', l' \rangle$$



Parameters & values:

- Exciton chemical potential: $\frac{\mu_X}{Ry^*}$
- Interlayer distance: $\frac{d}{a_B^*} = 1$
- Mass difference: $\delta_m \equiv \frac{m_e - m_h}{m_e + m_h} = 0$

Equivalence of two approaches

- First-order density matrix in dynamics equation approach

$$\begin{bmatrix} \rho_{cv\mathbf{k}}^{(1)}(\omega, \mathbf{q}) \\ \rho_{vc-\mathbf{k}}^{(1)}(\omega, \mathbf{q}) \end{bmatrix} = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}'} [\hbar\omega^+ \tau_z - \mathcal{H}(\mathbf{q})]_{\mathbf{k}, \mathbf{k}'}^{-1} \begin{bmatrix} o_{cv\mathbf{k}'}(-\mathbf{q}) \\ o_{vc-\mathbf{k}'}(-\mathbf{q}) \end{bmatrix} f(\omega, \mathbf{q})$$

- Polarization function in the Feynman diagram approach

$$(1 - \Pi^{\text{ir}} V^d) \Pi = \Pi^{\text{ir}} \implies \Pi^{-1} = [\Pi^{\text{ir}}]^{-1} - V^d$$

$$(1 + \Pi^0 V^x) \Pi^{\text{ir}} = \Pi^0 \implies [\Pi^{\text{ir}}]^{-1} = [\Pi^0]^{-1} + V^x$$

$$[\Pi(\omega)]_{ij,kl}^{-1} = [\Pi^0(i\nu_n \rightarrow \hbar\omega^+)]_{ij,kl}^{-1} - V_{ij,kl}^d + V_{ij,kl}^x \longrightarrow [\Pi(\omega)]^{-1} = \hbar\omega^+ \tau_z - \mathcal{H}$$

$(ij), (kl)$
taken from
 $(cv), (vc)$

The TDHF approximation from the dynamic equation approach is equivalent to including all the “ladder-bubble” diagram when calculating the two-particle correlation function

Exciton Meissner effect

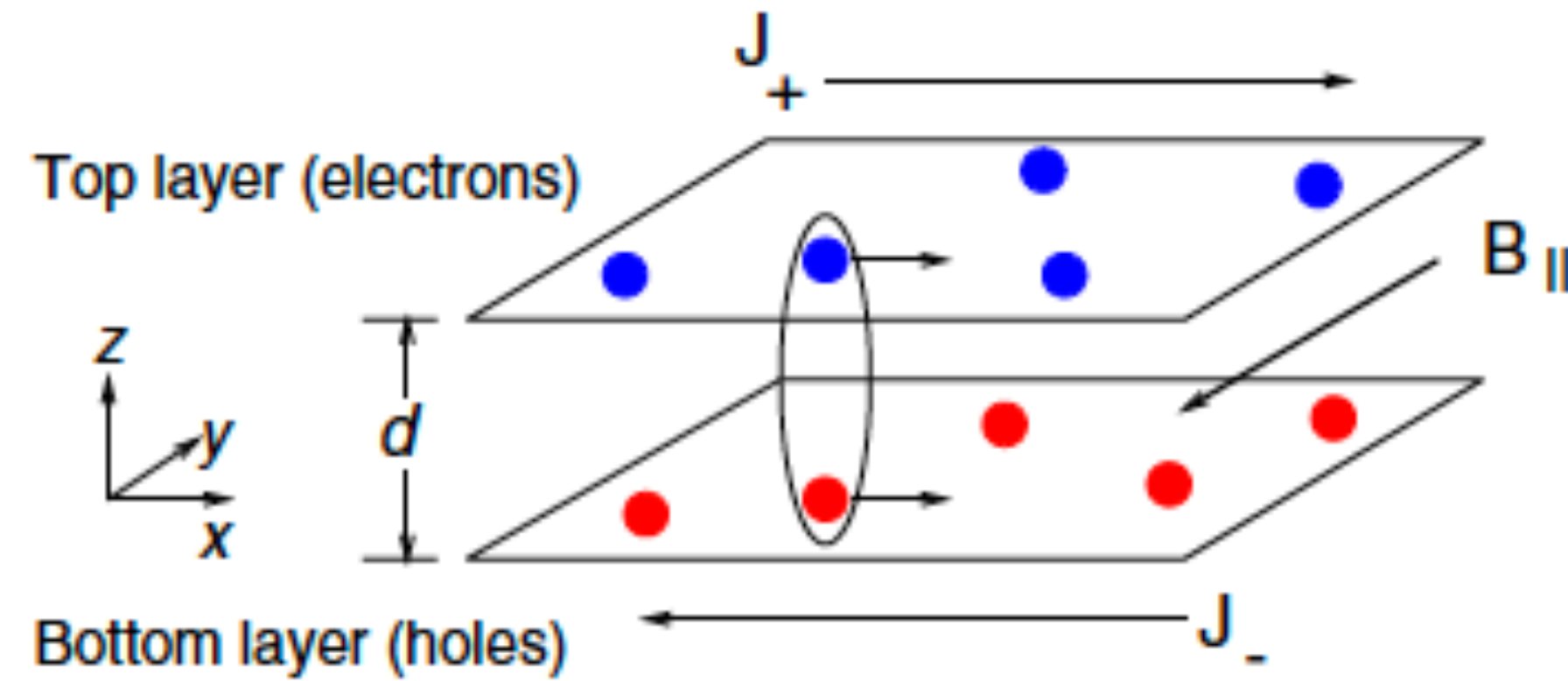


FIG. 1 (color online). Schematic bilayer electron-hole system. The electrons and holes form bound excitons which condense at low densities. We predict that an in-plane field $\mathbf{B}_{\parallel}$ will produce equal but opposite currents $\mathbf{J}_+$ and $\mathbf{J}_-$ in the two layers.

$$\mathbf{j}_X = \frac{e^2 \rho_X}{m} \mathbf{B}_t \times \hat{e}_z$$

Exciton London equation

Detecting the Goldstone mode via microwave impedance microscopy (MIM)

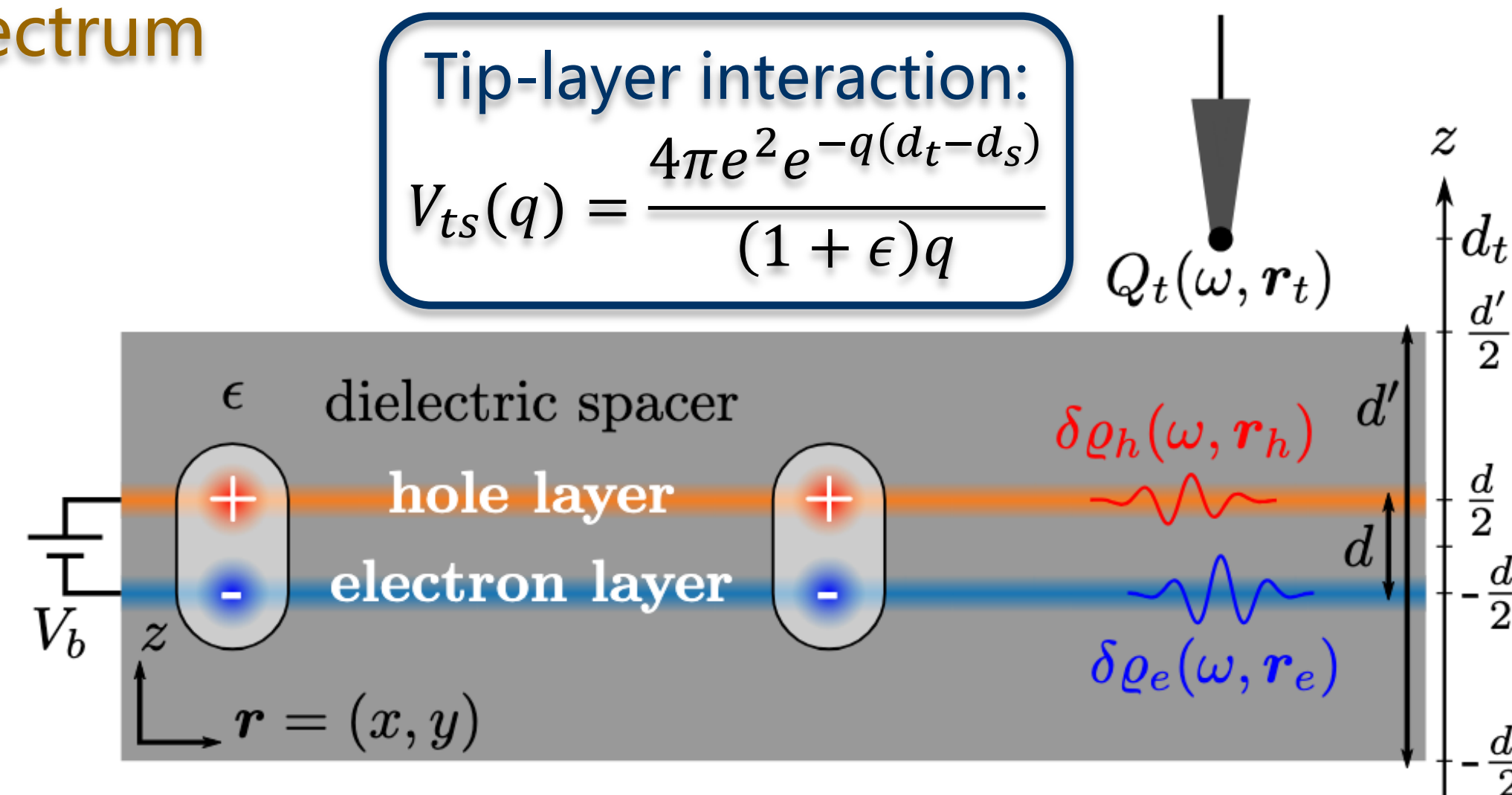
MIM spectrum

- Scalar potential induced by tip charge ($\sigma = \pm$ for symmetric and anti-symmetric)

$$\phi^{\sigma=\pm}(\omega, \mathbf{r}) = \tilde{V}^{\sigma=\pm}(\mathbf{r} - \mathbf{r}_t) Q_t(\omega)$$

$$\tilde{V}^+(\mathbf{r}) = e^{-2}[V_{te}(\mathbf{r}) + V_{th}(\mathbf{r})]/2$$

$$\tilde{V}^{-}(\mathbf{r}) = e^{-2}[V_{te}(\mathbf{r}) - V_{th}(\mathbf{r})]$$



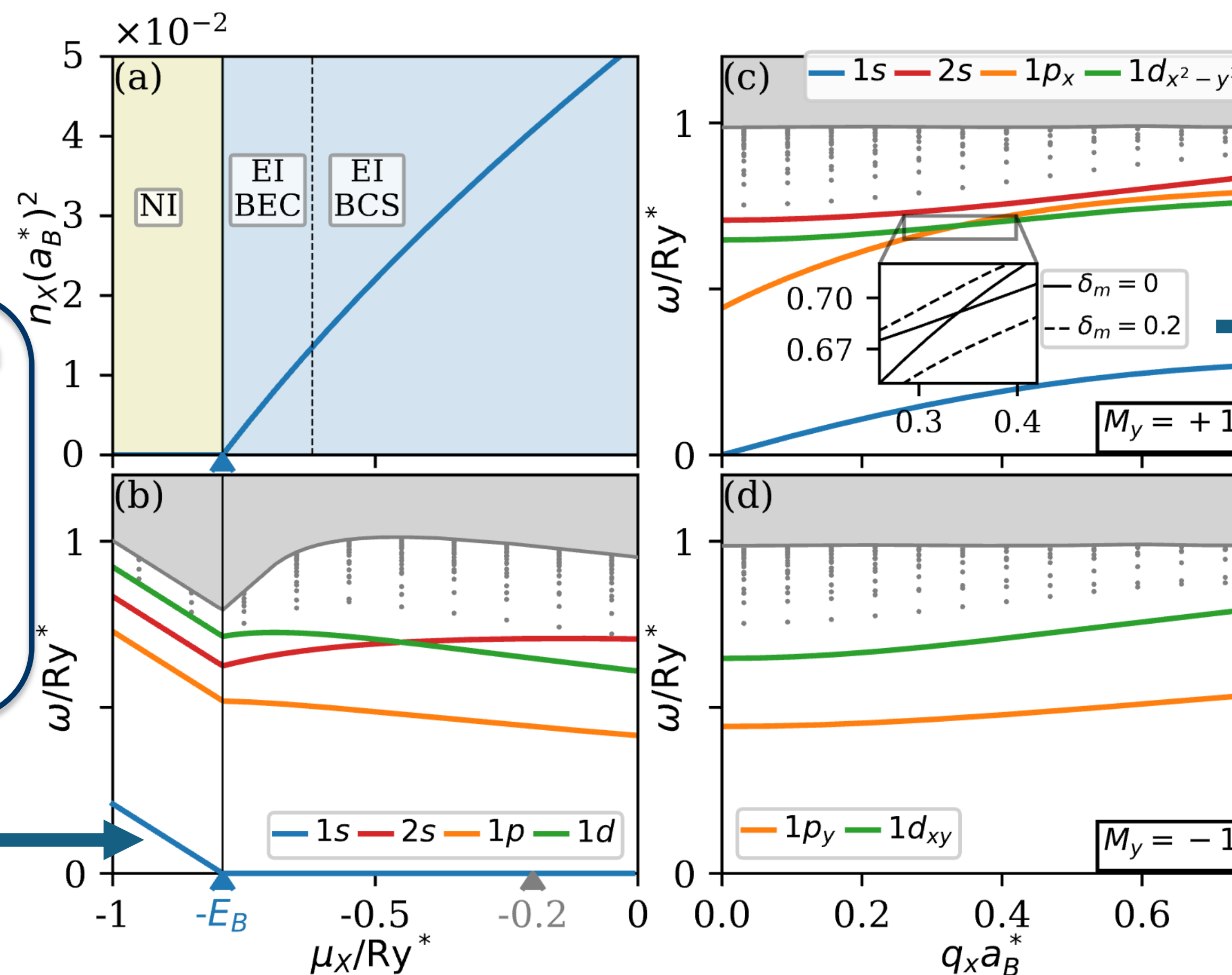
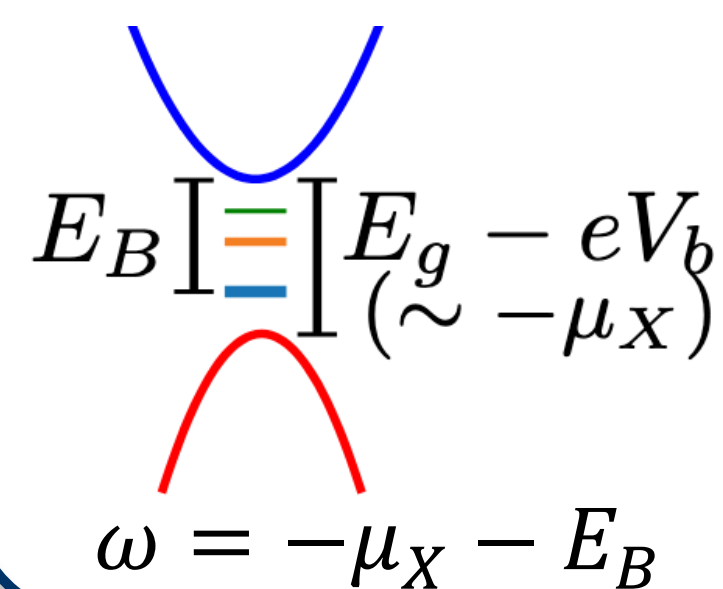
- Charge and exciton density fluctuations ($\sigma = \pm$ for charge and exciton)

$$\delta \varrho^\sigma(\omega, \mathbf{r}) = - \int d\mathbf{r}' K_{00}^\sigma(\omega, \mathbf{r} - \mathbf{r}') \phi^\sigma(\omega, \mathbf{r}') = -\mathcal{V}^{-1} \sum_{\mathbf{q}} K_{00}^\sigma(\omega, \mathbf{q}) \tilde{V}^\sigma(\mathbf{q}) Q_t(\omega) e^{i\mathbf{q} \cdot (\mathbf{r} - \mathbf{r}_t)}$$

- Potential feedback to the tip

$$\delta U(\omega) = \sum_{\sigma=\pm} \int d\mathbf{r} \tilde{V}^\sigma(\mathbf{r}_t - \mathbf{r}) \delta \varrho^\sigma(\omega, \mathbf{r}) = -\mathcal{V}^{-1} \sum_{\sigma=\pm, \mathbf{q}} [\tilde{V}^\sigma(\mathbf{q})]^{-2} K_{00}^\sigma(\omega, \mathbf{q}) Q_t(\omega)$$

Inter-layer exciton



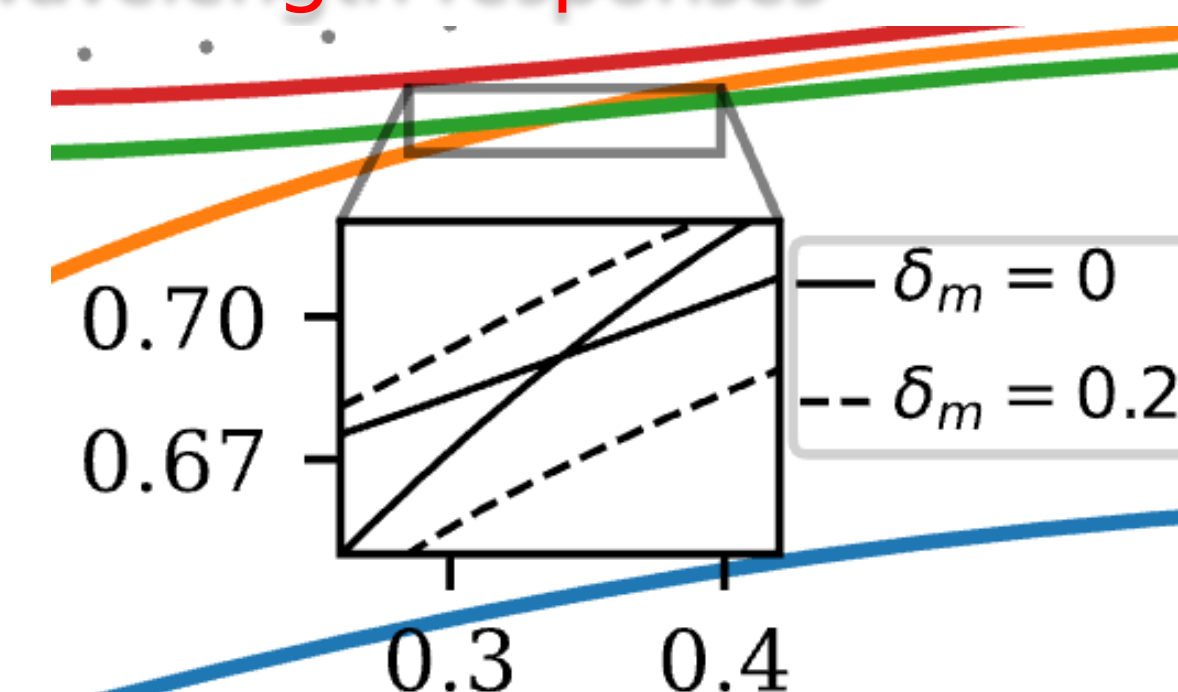
(a) Phase diagram
 (b) Excitation spectrum at $q = 0$
 (c)(d) Spectrum at $\mu_X = -0.2$ along q_x axis with different M_y eigenvalues

Parameters & values:

- Exciton chemical potential: $\frac{\mu_X}{Ry^*}$
- Interlayer distance: $\frac{d}{a_B^*} = 1$
- Mass difference: $\delta_m \equiv \frac{m_e - m_h}{m_e + m_h} = 0$

Effect of particle-hole asymmetry

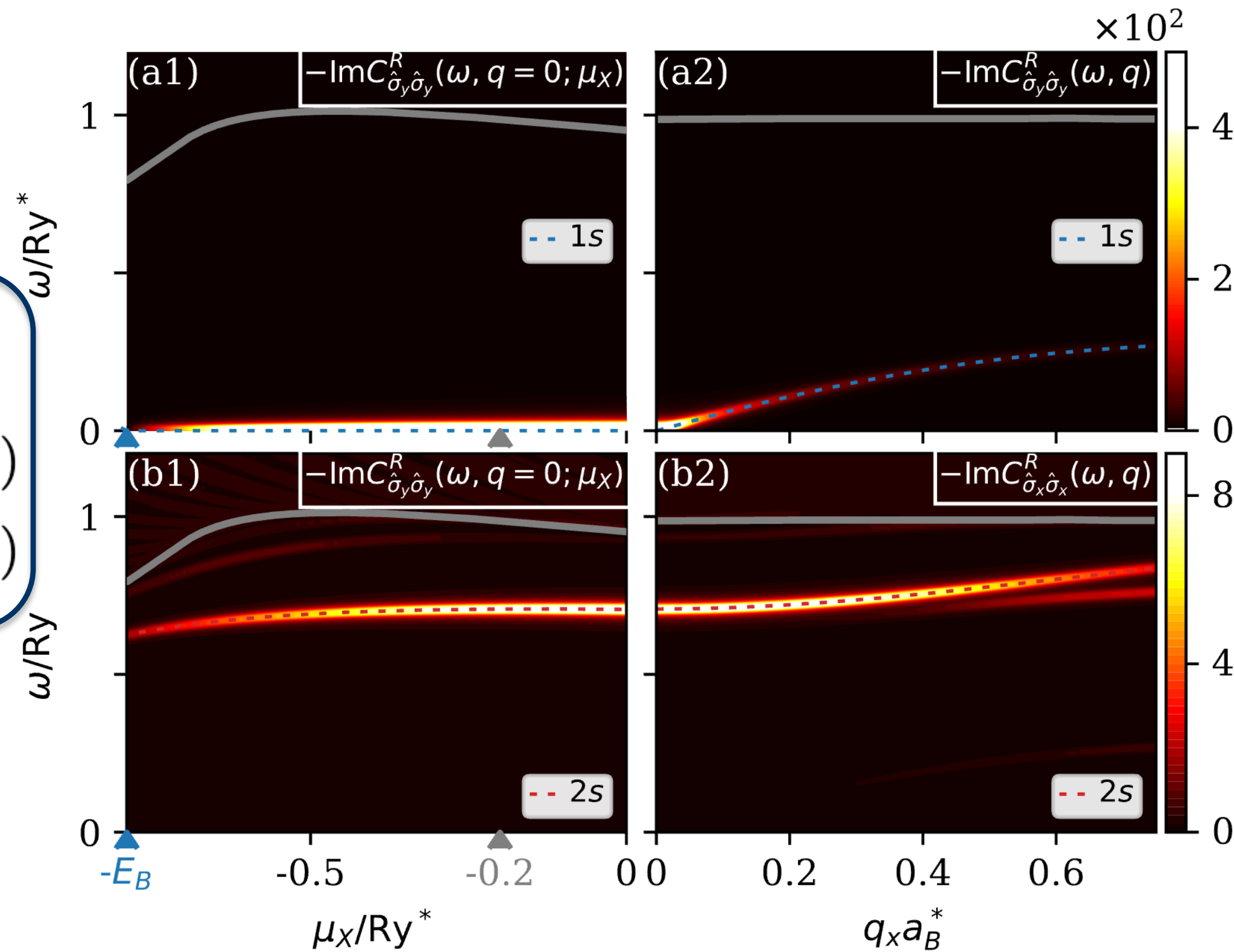
- Open a gap for the level crossing protected by particle-hole symmetry
- Insignificant away from the level crossing for typical value for TMD bilayers $\delta_m = 0.2$
- Ignored when studying long-wavelength responses



Phase and amplitude
fluctuation operators

$$\hat{\sigma}_y(\mathbf{r}) = \Psi^\dagger(\mathbf{r})\sigma_y\Psi(\mathbf{r})$$

$$\hat{\sigma}_x(\mathbf{r}) = \Psi^\dagger(\mathbf{r})\sigma_x\Psi(\mathbf{r})$$



The phase and amplitude modes
correspond to the
1s and 2s monopole modes.

(a1)(a2) Correlation function of phase fluctuation operator σ_y . (a1) $q = 0$; (a2) $\mu_x = -0.2$.
(b1)(b2) Correlation function of amplitude fluctuation operator σ_x . (b1) $q = 0$; (b2) $\mu_x = -0.2$.

Ground state: Hartree Fock method

$$\hat{H}_{MF} = \sum_{\mathbf{k}} [c_{e\mathbf{k}}^\dagger, c_{h\mathbf{k}}^\dagger] \begin{bmatrix} \zeta_{\mathbf{k}} + \varepsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & \zeta_{\mathbf{k}} - \varepsilon_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{e\mathbf{k}} \\ c_{h\mathbf{k}} \end{bmatrix}$$

Quasi-particle creation operators

$$\begin{aligned} c_{v\mathbf{k}}^\dagger &\equiv \alpha_{\mathbf{k}} c_{e\mathbf{k}}^\dagger + \beta_{\mathbf{k}} c_{h\mathbf{k}}^\dagger \\ c_{c\mathbf{k}}^\dagger &\equiv \beta_{\mathbf{k}} c_{e\mathbf{k}}^\dagger - \alpha_{\mathbf{k}} c_{h\mathbf{k}}^\dagger \end{aligned}$$

$$\alpha_{\mathbf{k}} = \sqrt{(1 - \varepsilon_{\mathbf{k}}/\xi_{\mathbf{k}})/2}$$

$$\beta_{\mathbf{k}} = \sqrt{(1 + \varepsilon_{\mathbf{k}}/\xi_{\mathbf{k}})/2}$$

$$\xi_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$$

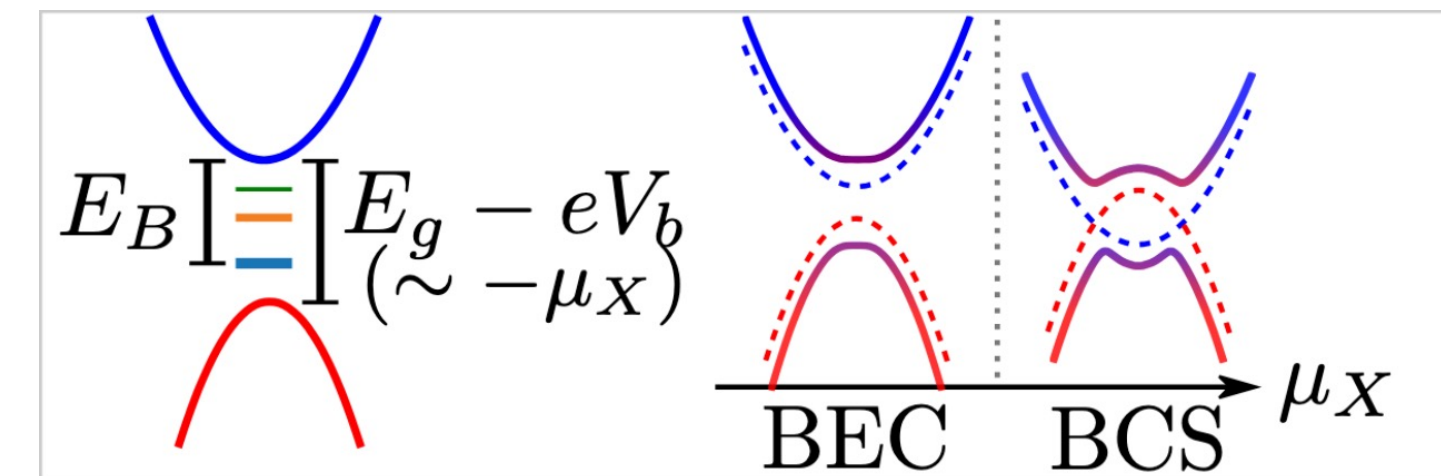
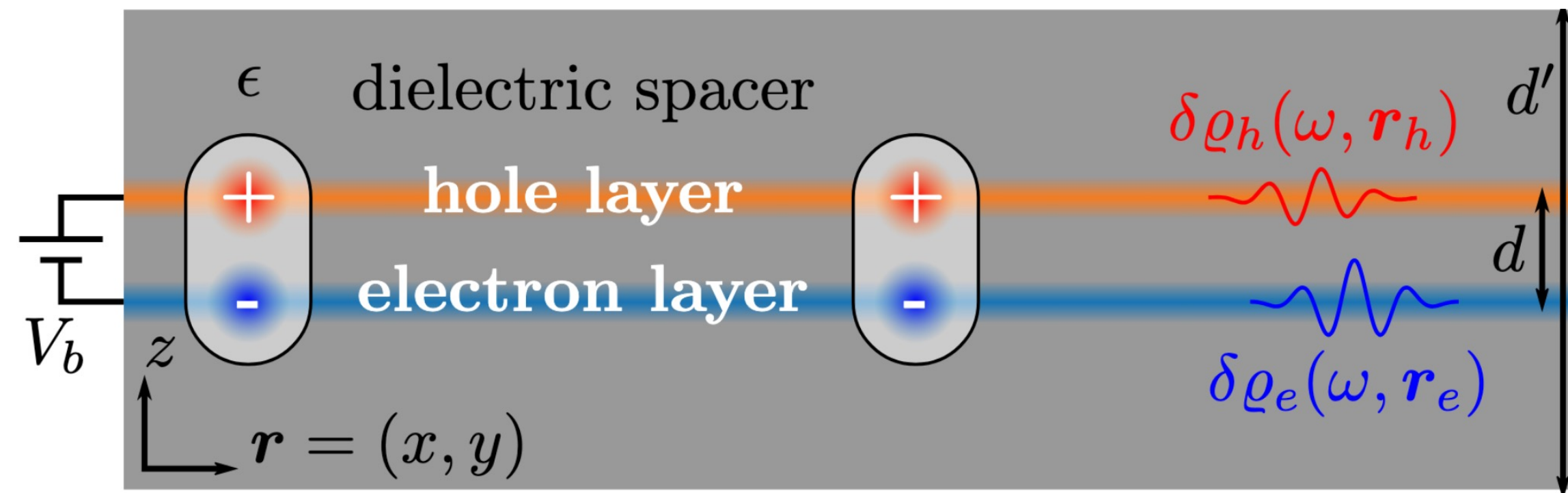
$$\hat{H}_{MF} = \sum_{\mathbf{k}} [c_{c\mathbf{k}}^\dagger, c_{v\mathbf{k}}^\dagger] \begin{bmatrix} \zeta_{\mathbf{k}} + \xi_{\mathbf{k}} & 0 \\ 0 & \zeta_{\mathbf{k}} - \xi_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{c\mathbf{k}} \\ c_{v\mathbf{k}} \end{bmatrix}$$

Single particle density matrix

$$\rho_{ee\mathbf{k}}^X = \alpha_{\mathbf{k}}^2 = \frac{1}{2} \left(1 - \frac{\varepsilon_{\mathbf{k}}}{\xi_{\mathbf{k}}} \right)$$

$$\rho_{eh\mathbf{k}}^X = \alpha_{\mathbf{k}} \beta_{\mathbf{k}} = -\frac{\Delta_{\mathbf{k}}}{2\xi_{\mathbf{k}}}$$

Two-dimensional interacting electron-hole gases model



Many-body Hamiltonian

$$\hat{H}_0 = \sum_{\mathbf{k}} [c_{e\mathbf{k}}^\dagger, c_{h\mathbf{k}}^\dagger] \begin{bmatrix} \frac{\hbar^2 k^2}{2m_e} - \frac{\mu_X}{2} & 0 \\ 0 & -\frac{\hbar^2 k^2}{2m_h} + \frac{\mu_X}{2} \end{bmatrix} \begin{bmatrix} c_{e\mathbf{k}} \\ c_{h\mathbf{k}} \end{bmatrix},$$

$$\hat{H}_I = \frac{1}{2\mathcal{V}} \sum_{ss'=eh} \sum_{\mathbf{k}\mathbf{k}'\mathbf{q}} V_{ss'}(\mathbf{q}) c_{s\mathbf{k}}^\dagger c_{s'\mathbf{k}'}^\dagger c_{s'\mathbf{k}'+\mathbf{q}} c_{s\mathbf{k}-\mathbf{q}},$$

Coulomb interaction

$$V(\mathbf{q}) \equiv V_{s=s'}(\mathbf{q}) = 2\pi e^2 / \epsilon q$$

$$U(\mathbf{q}) \equiv V_{s \neq s'} = V(\mathbf{q}) e^{-qd}$$

$$d' \gg d$$

Mean-field Hamiltonian

$$\hat{H}_{MF} = \sum_{\mathbf{k}} [c_{e\mathbf{k}}^\dagger, c_{h\mathbf{k}}^\dagger] \begin{bmatrix} \zeta_{\mathbf{k}} + \varepsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & \zeta_{\mathbf{k}} - \varepsilon_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{e\mathbf{k}} \\ c_{h\mathbf{k}} \end{bmatrix}$$

- Electron-hole asymmetry

$$\zeta_{\mathbf{k}} \equiv \hbar^2 k^2 \delta_m / 4m$$

- Renormalized band energy difference

$$\varepsilon_{\mathbf{k}} \equiv \frac{\hbar^2 k^2}{4m} - \frac{\mu_X}{2} + \frac{2\pi e^2 d n_X}{\epsilon} - \frac{1}{\mathcal{V}} \sum_{\mathbf{k}'} V(\mathbf{k} - \mathbf{k}') \rho_{ee\mathbf{k}'}^X$$

- Inter-layer coherence

$$\Delta_{\mathbf{k}} \equiv -\frac{1}{\mathcal{V}} \sum_{\mathbf{k}'} U(\mathbf{k} - \mathbf{k}') \rho_{eh\mathbf{k}'}^X$$

arg $\Delta_{\mathbf{k}}$ has a phase freedom
chosen to be real and negative

Excitonic units

- Reduced mass: $m = \frac{m_e m_h}{m_e + m_h}$
- Length unit: $a_B^* = \frac{\epsilon \hbar^2}{m^* e^2}$
- Energy unit: $Ry^* = \frac{e^2}{2\epsilon a_B^*}$

Independent parameters

- Exciton chemical potential: $\frac{\mu_{ex}}{Ry^*}$
- Interlayer distance: $\frac{d}{a_B^*}$
- Mass difference: $\delta_m \equiv \frac{m_e - m_h}{m_e + m_h}$

Single particle density matrix

$$\rho_{ss'\mathbf{k}}^X \equiv \langle c_{s'\mathbf{k}}^\dagger c_{s\mathbf{k}} \rangle$$

Mean-field Hamiltonian

$$\hat{H}_{MF} = \sum_{\mathbf{k}} [c_{e\mathbf{k}}^\dagger, c_{h\mathbf{k}}^\dagger] \begin{bmatrix} \zeta_{\mathbf{k}} + \varepsilon_{\mathbf{k}} & \Delta_{\mathbf{k}} \\ \Delta_{\mathbf{k}}^* & \zeta_{\mathbf{k}} - \varepsilon_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{e\mathbf{k}} \\ c_{h\mathbf{k}} \end{bmatrix}$$

Quasi-particle

$$\begin{aligned} c_{v\mathbf{k}}^\dagger &\equiv \alpha_{\mathbf{k}} c_{e\mathbf{k}}^\dagger + \beta_{\mathbf{k}} c_{h\mathbf{k}}^\dagger \\ c_{c\mathbf{k}}^\dagger &\equiv \beta_{\mathbf{k}} c_{e\mathbf{k}}^\dagger - \alpha_{\mathbf{k}} c_{h\mathbf{k}}^\dagger \end{aligned}$$

$$\begin{aligned} \alpha_{\mathbf{k}} &= \sqrt{(1 - \varepsilon_{\mathbf{k}}/\xi_{\mathbf{k}})/2} \\ \beta_{\mathbf{k}} &= \sqrt{(1 + \varepsilon_{\mathbf{k}}/\xi_{\mathbf{k}})/2} \\ \xi_{\mathbf{k}} &= \sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2} \end{aligned}$$

$$\hat{H}_{MF} = \sum_{\mathbf{k}} [c_{c\mathbf{k}}^\dagger, c_{v\mathbf{k}}^\dagger] \begin{bmatrix} \zeta_{\mathbf{k}} + \xi_{\mathbf{k}} & 0 \\ 0 & \zeta_{\mathbf{k}} - \xi_{\mathbf{k}} \end{bmatrix} \begin{bmatrix} c_{c\mathbf{k}} \\ c_{v\mathbf{k}} \end{bmatrix}$$

$$\begin{aligned} \rho_{ee\mathbf{k}}^X &= \alpha_{\mathbf{k}}^2 = \frac{1}{2} \left(1 - \frac{\varepsilon_{\mathbf{k}}}{\xi_{\mathbf{k}}} \right) \\ \rho_{eh\mathbf{k}}^X &= \alpha_{\mathbf{k}} \beta_{\mathbf{k}} = -\frac{\Delta_{\mathbf{k}}}{2\xi_{\mathbf{k}}} \end{aligned}$$

Excitonic units

- Reduced mass: $m = \frac{m_e m_h}{m_e + m_h}$
- Length unit: $a_B^* = \frac{\epsilon \hbar^2}{m^* e^2}$
- Energy unit: $Ry^* = \frac{e^2}{2\epsilon a_B^*}$

Independent parameters

- Exciton chemical potential: $\frac{\mu_{ex}}{Ry^*}$
- Interlayer distance: $\frac{d}{a_B^*}$
- Mass difference: $\delta_m \equiv \frac{m_e - m_h}{m_e + m_h}$

Single particle density matrix

$$\rho_{ss'\mathbf{k}}^X \equiv \langle c_{s'\mathbf{k}}^\dagger c_{s\mathbf{k}} \rangle$$

Collective modes: time-dependent Hartree Fock

1. Coupling to external field

$$\hat{H}_c = \frac{1}{\mathcal{V}} \sum_{\mathbf{a}} f(t, \mathbf{q}) \hat{O}(-\mathbf{q})$$

$$\hat{O}(\mathbf{q}) = \sum_{i,j=c,v} \sum_{\mathbf{k}} o_{ij\mathbf{k}}(\mathbf{q}) c_{i\mathbf{k}-\mathbf{q}/2}^\dagger c_{j\mathbf{k}+\mathbf{q}/2}$$

2. Dynamic equation of the single particle density matrix

$$\rho_{ij\mathbf{k}}(t, \mathbf{q}) \equiv \langle c_{j\mathbf{k}-\mathbf{q}/2}^\dagger c_{i\mathbf{k}+\mathbf{q}/2} \rangle$$

$$i\hbar \partial_t \rho_{ij\mathbf{k}}(t, \mathbf{q}) = \langle [c_{j\mathbf{k}-\mathbf{q}/2}^\dagger c_{i\mathbf{k}+\mathbf{q}/2}, \hat{H}_0 + \hat{H}_I + \hat{H}_c] \rangle$$

3. Linearization of the dynamic equation

$$\rho_{ij\mathbf{k}}(t, \mathbf{q}) = \sum \rho_{ij\mathbf{k}}^{(n)}(t, \mathbf{q})$$

$$\rho_{ij\mathbf{k}}^{(0)}(t, \mathbf{q}) = \rho_{ij\mathbf{k}}^X \delta_{\mathbf{q}0} = \delta_{ij} \delta_{iv} \delta_{\mathbf{q}0}$$

$$i\hbar \tau_z \partial_t \begin{bmatrix} \rho_{cv\mathbf{k}}^{(1)}(t, \mathbf{q}) \\ \rho_{vc-\mathbf{k}}^{(1)}(t, \mathbf{q}) \end{bmatrix} = \sum_{\mathbf{k}'} \mathcal{H}_{\mathbf{k},\mathbf{k}'}(\mathbf{q}) \begin{bmatrix} \rho_{cv\mathbf{k}}^{(1)}(t, \mathbf{q}) \\ \rho_{vc-\mathbf{k}}^{(1)}(t, \mathbf{q}) \end{bmatrix} + \frac{1}{\mathcal{V}} \begin{bmatrix} o_{cv\mathbf{k}}(-\mathbf{q}) \\ o_{vc-\mathbf{k}}(-\mathbf{q}) \end{bmatrix} f(t, \mathbf{q})$$

4. Collective modes and excitation energy

$$\sum_{\mathbf{k}'} \mathcal{H}_{\mathbf{k},\mathbf{k}'}(\mathbf{q}) \Phi_{n\mathbf{k}'}(\mathbf{q}) = \hbar \omega_n(\mathbf{q}) \tau_z \Phi_{n\mathbf{k}}(\mathbf{q})$$

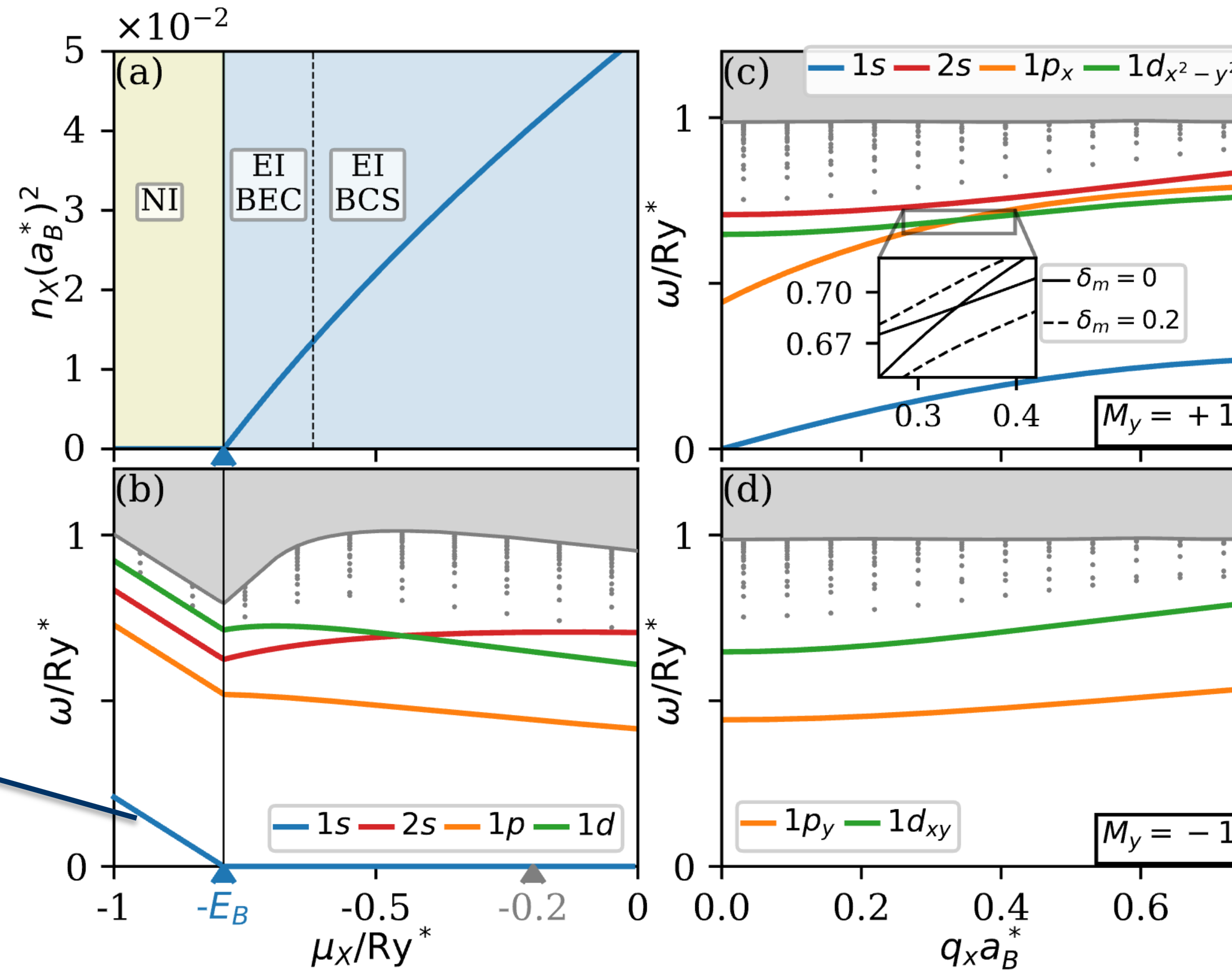
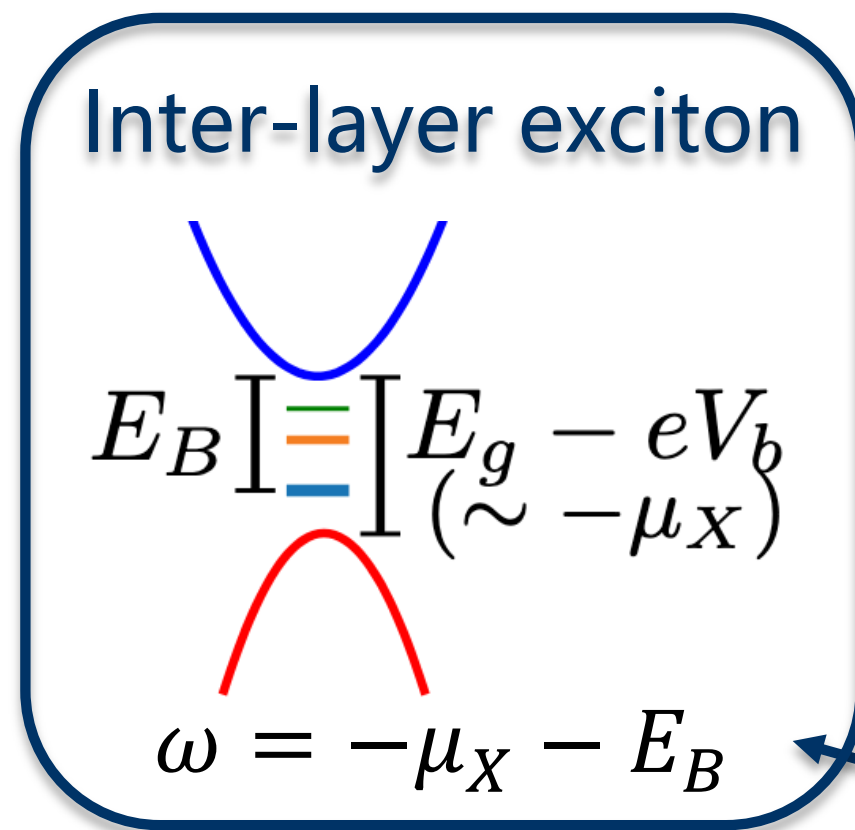
5. Correlation function

$$\langle \hat{O}' \rangle(\omega, \mathbf{q}) = \sum_{ij\mathbf{k}} o'_{ij\mathbf{k}}(\mathbf{q}) \rho_{jik}^{(1)}(\omega, \mathbf{q}) = C_{\hat{O}', \hat{O}}^R(\omega, \mathbf{q}) f(\omega, \mathbf{q})$$

$$C_{\hat{O}', \hat{O}}^R(\omega, \mathbf{q}) = \frac{1}{\mathcal{V}} \sum_n \frac{\omega_n(\mathbf{q}) [O'_n(\mathbf{q})]^* O_n(\mathbf{q})}{\omega + i\eta - \omega_n(\mathbf{q})}$$

$$O_n(\mathbf{q}) \equiv \sum_{\mathbf{k}} \Phi_{n\mathbf{k}}^\dagger(\mathbf{q}) \begin{bmatrix} o_{cv\mathbf{k}}(-\mathbf{q}) \\ o_{vc-\mathbf{k}}(\mathbf{q}) \end{bmatrix}$$

Results: the collective modes spectrum



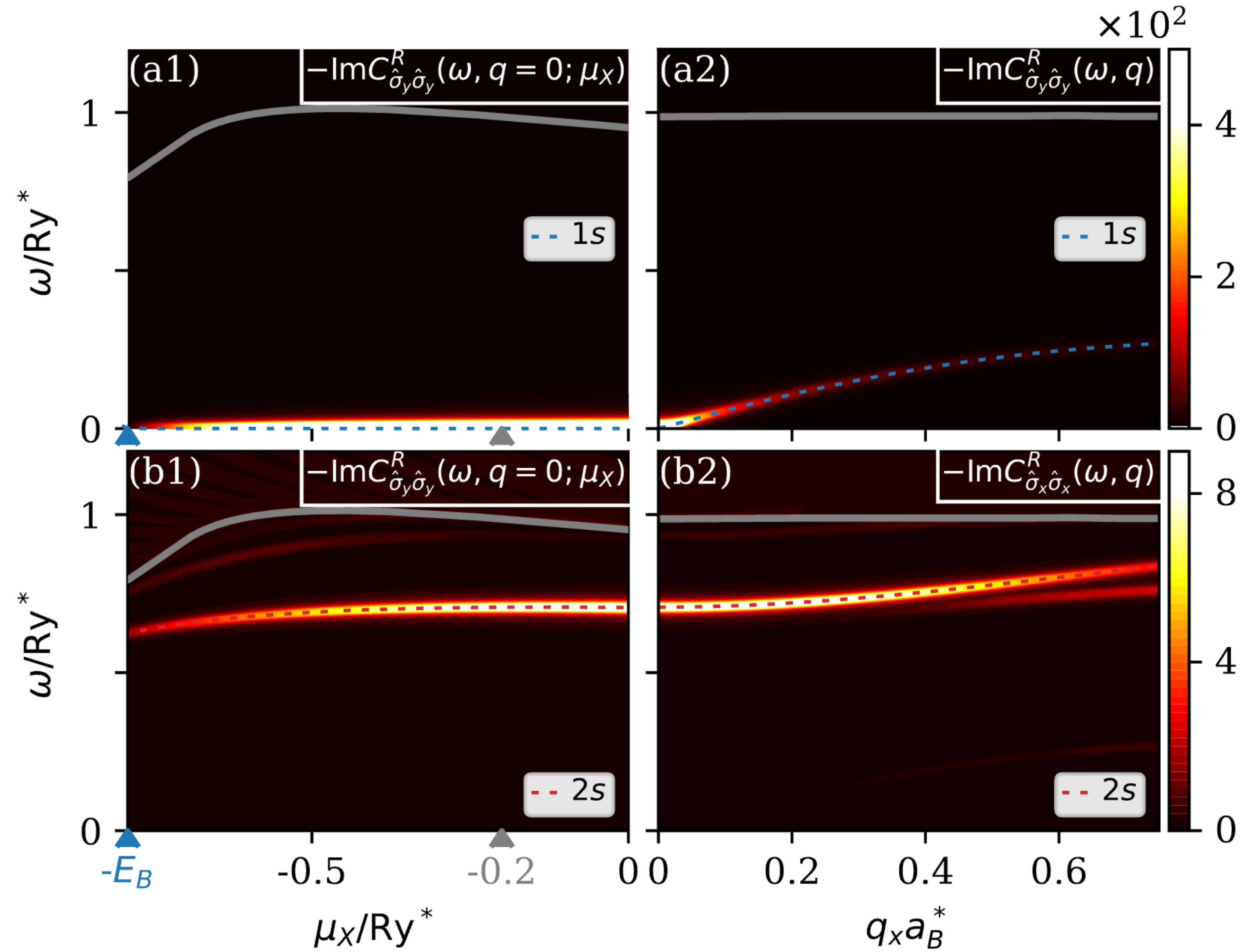
- Parameter values:
- Exciton chemical potential: $\frac{\mu_X}{Ry^*}$
 - Interlayer distance: $\frac{d}{a_B^*} = 1$
 - Mass difference: $\delta_m \equiv \frac{m_e - m_h}{m_e + m_h} = 0$

(a) Phase diagram
 (b) Excitation spectrum at $q = 0$
 (c)(d) Spectrum at $\mu_X = -0.2$ along q_x axis with different M_y eigenvalues
 Inset of (c) Spectrum near the level crossing point with small mass difference

Results: Phase and amplitude modes

Phase and amplitude fluctuation operators

$$\hat{\sigma}_y(\mathbf{r}) = \Psi^\dagger(\mathbf{r})\sigma_y\Psi(\mathbf{r})$$

$$\hat{\sigma}_x(\mathbf{r}) = \Psi^\dagger(\mathbf{r})\sigma_x\Psi(\mathbf{r})$$


(a1)(a2) Correlation function of phase fluctuation operator σ_y . (a1) $q = 0$; (a2) $\mu_x = -0.2$.
 (b1)(b2) Correlation function of amplitude fluctuation operator σ_x . (b1) $q = 0$; (b2) $\mu_x = -0.2$.

Results: response to layer symmetric gauge field

Coupling to layer symmetric gauge field

$$A_{e\mu}(t, \mathbf{r}) = A_{h\mu}(t, \mathbf{r}) = A_{\mu}^{+}(t, \mathbf{r}) = (\phi^{+}(t, \mathbf{r}), \mathbf{A}^{+}(t, \mathbf{r}))$$

$$\hat{H}_c = \int d\mathbf{r} [\hat{\varrho}^{+}(\mathbf{r})\phi^{+}(t, \mathbf{r}) - \hat{\mathbf{j}}_p^{+}(\mathbf{r}) \cdot \mathbf{A}^{+}(t, \mathbf{r})]$$

Charge density and current operators

$$\hat{\varrho}^{+}(\mathbf{r}) = -e\Psi^{\dagger}(\mathbf{r})\Psi(\mathbf{r}),$$

$$\hat{\mathbf{j}}_p^{+}(\mathbf{r}) = \frac{ie\hbar}{4m}\Psi^{\dagger}(\mathbf{r})\sigma_z\nabla_r\Psi(\mathbf{r}) + h.c.,$$

$$\hat{\mathbf{j}}_d^{+}(t, \mathbf{r}) = -\frac{e}{2m}\Psi^{\dagger}(\mathbf{r})\sigma_z\Psi(\mathbf{r})\mathbf{A}^{+}(t, \mathbf{r})$$

$$\hat{\mathbf{j}}_{\mu}^{+}(t, \mathbf{r}) = (-\hat{\varrho}^{+}(\mathbf{r}), \hat{\mathbf{j}}^{+}(t, \mathbf{r}))$$

Response functions

$$j_{\mu}^{+}(\omega, \mathbf{q}) = \sum_{\nu} K_{\mu\nu}^{+}(\omega, \mathbf{q}) A_{\nu}^{+}(\omega, \mathbf{q})$$

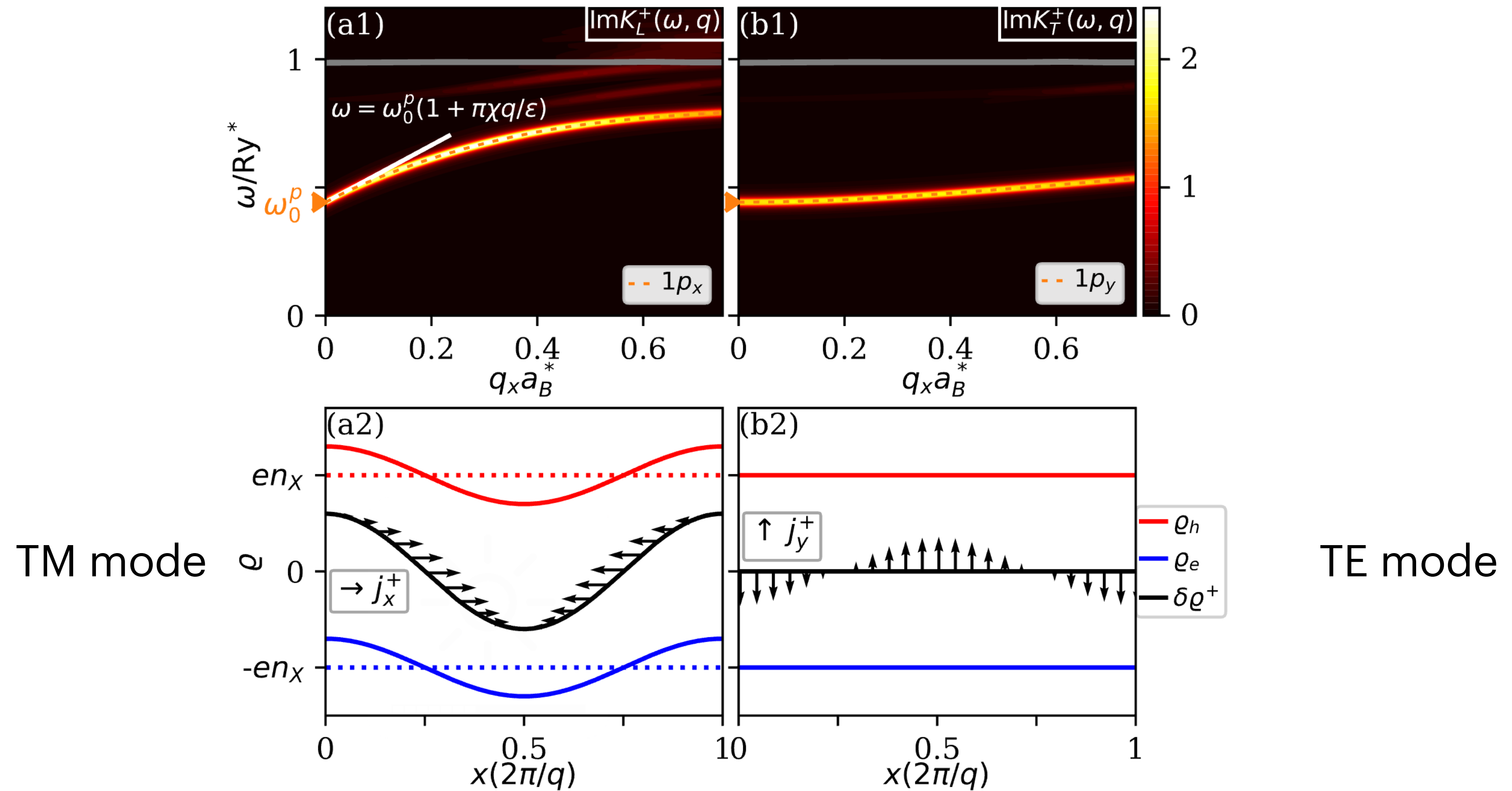
$$K_{\mu\nu}^{+}(\omega, \mathbf{q}) = -(1 - \delta_{\mu 0})\delta_{\mu\nu} \frac{en_X}{m} - C_{\hat{j}_{p\mu}^{+}\hat{j}_{p\nu}^{+}}^R(\omega, \mathbf{q})$$

$$\sum_{\mu} q_{\mu} K_{\mu\nu}^{+}(\omega, \mathbf{q}) = \sum_{\nu} K_{\mu\nu}^{+}(\omega, \mathbf{q}) q_{\nu} = 0$$

$$K_{ab}^{+}(\omega, \mathbf{q}) = K_L^{+}(\omega, \mathbf{q}) \frac{q_a q_b}{|\mathbf{q}|^2} + K_T^{+}(\omega, \mathbf{q}) \left(\delta_{ab} - \frac{q_a q_b}{|\mathbf{q}|^2} \right)$$

Due to **Ward identity** (charge conservation) and **rotation symmetry**, only **2 independent** components in the 9 responses functions

Results: response to layer symmetric gauge field



(a1)(a2) Charge density and longitudinal current fluctuations
 (b1)(b2) Transverse current fluctuations

Results: response to layer anti-symmetric gauge field

Coupling to layer anti-symmetric gauge field

$$A_{e\mu}(t, \mathbf{r}) = -A_{h\mu}(t, \mathbf{r}) = A_{\mu}^{-}(t, \mathbf{r})/2$$

$$\hat{H}_c = \int d\mathbf{r} [\hat{\varrho}^{-}(\mathbf{r})\phi^{-}(t, \mathbf{r}) - \hat{\mathbf{j}}_p^{-}(\mathbf{r}) \cdot \mathbf{A}^{-}(t, \mathbf{r})]$$

Exciton density and current operators

$$\hat{\varrho}^{-}(\mathbf{r}) = -\frac{e}{2}\Psi^{\dagger}(\mathbf{r})\sigma_z\Psi(\mathbf{r})$$

$$\hat{\mathbf{j}}_p^{-}(\mathbf{r}) = \frac{ie\hbar}{4m}\Psi^{\dagger}(\mathbf{r})\nabla_{\mathbf{r}}\Psi(\mathbf{r}) + h.c.$$

$$\hat{\mathbf{j}}_d^{-}(t, \mathbf{r}) = -\frac{e}{4m}\Psi^{\dagger}(\mathbf{r})\sigma_z\Psi(\mathbf{r})\mathbf{A}^{-}(t, \mathbf{r})$$

$$\hat{\mathbf{j}}_{\mu}^{-}(t, \mathbf{r}) = (-\hat{\varrho}^{-}(\mathbf{r}), \hat{\mathbf{j}}^{-}(t, \mathbf{r}))$$

Response functions

$$j_{\mu}^{-}(\omega, \mathbf{q}) \equiv \langle \hat{j}_{\mu}^{-}(\omega, \mathbf{q}) \rangle = \sum_{\nu} K_{\mu\nu}^{-}(\omega, \mathbf{q}) A_{\nu}^{-}(\omega, \mathbf{q})$$

$$K_{\mu\nu}^{-}(\omega, \mathbf{q}) = -(1 - \delta_{\mu 0})\delta_{\mu\nu} \frac{en_X}{2m} - C_{\hat{j}_{p\mu}^{-}\hat{j}_{p\nu}^{-}}^R(\omega, \mathbf{q})$$

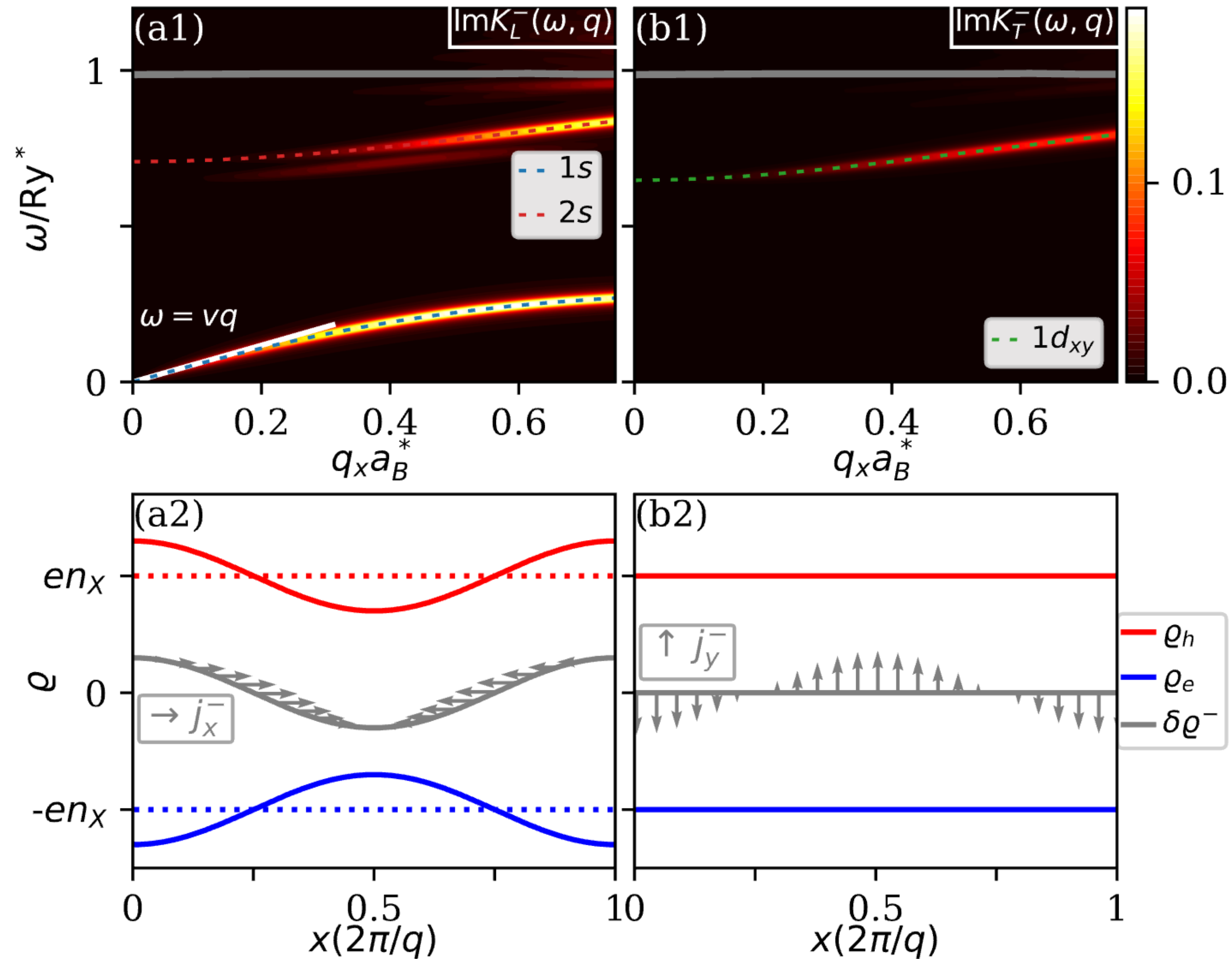
$$\sum_{\mu} q_{\mu} K_{\mu\nu}^{-}(\omega, \mathbf{q}) = \sum_{\nu} K_{\mu\nu}^{-}(\omega, \mathbf{q}) q_{\nu} = 0$$

$$K_{ab}^{-}(\omega, \mathbf{q}) = K_L^{-}(\omega, \mathbf{q}) \frac{q_a q_b}{|\mathbf{q}|^2} + K_T^{-}(\omega, \mathbf{q}) \left(\delta_{ab} - \frac{q_a q_b}{|\mathbf{q}|^2} \right)$$

Due to **Ward identity** (charge conservation) and **rotation symmetry**, only **2 independent** components in the 9 responses functions

Results: response to layer anti-symmetric gauge field

TM mode



TE mode

(a1)(a2) Exciton density and longitudinal current fluctuations
 (b1)(b2) Transverse current fluctuations

Results: detecting the Goldstone mode via microwave impedance microscopy

Layer symmetric and anti-symmetric scalar potential induced by tip charge

$$\phi^\pm(\omega, \mathbf{r}) = g^\pm(\mathbf{r} - \mathbf{r}_t)Q_t(\omega)$$

$$g^+(\mathbf{r}) = e^{-2}[V_{te}(\mathbf{r}) + V_{th}(\mathbf{r})]/2$$

$$g^-(\mathbf{r}) = e^{-2}[V_{te}(\mathbf{r}) - V_{th}(\mathbf{r})]$$

Charge and exciton density fluctuations

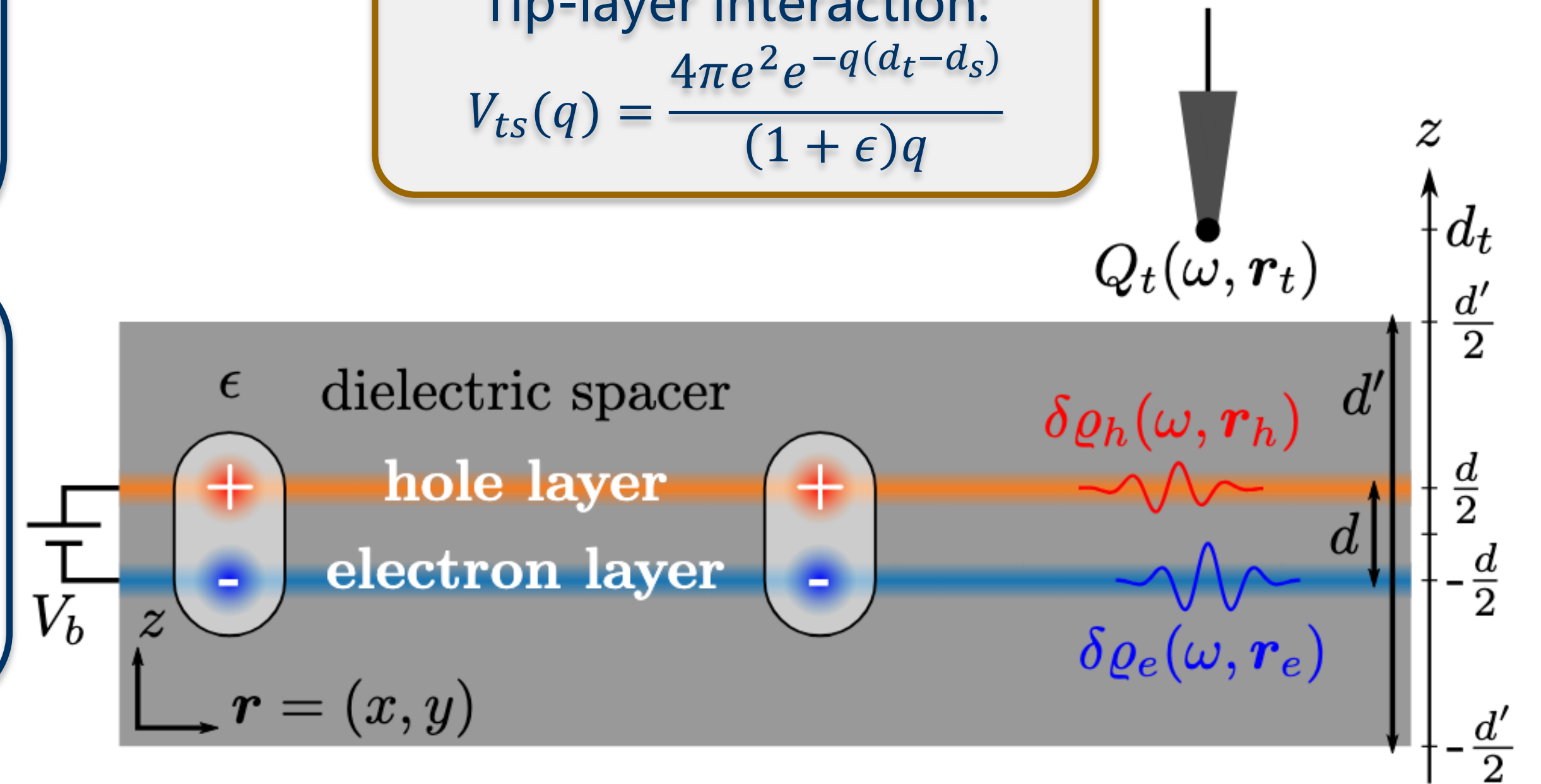
$$\begin{aligned} \delta\rho^\pm(\omega, \mathbf{r}) &= - \int d\mathbf{r}' K_{00}^\pm(\omega, \mathbf{r} - \mathbf{r}')\phi^\pm(\omega, \mathbf{r}') \\ &= - \mathcal{V}^{-1} \sum_{\mathbf{q}} K_{00}^\pm(\omega, \mathbf{q})g^\pm(\mathbf{q})Q_t(\omega)e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}_t)} \end{aligned}$$

Potential feedback to the tip

$$\begin{aligned} \delta U(\omega) &= \sum_{s=eh} \int d\mathbf{r} e^{-2}V_{ts}(\mathbf{r}_t - \mathbf{r})\delta\rho_s(\omega, \mathbf{r}) \\ &= \sum_{\sigma=\pm} \int d\mathbf{r} g^\sigma(\mathbf{r}_t - \mathbf{r})\delta\rho^\sigma(\omega, \mathbf{r}) \\ &= - \mathcal{V}^{-1} \sum_{\sigma=\pm, \mathbf{q}} [g^\sigma(\mathbf{q})]^{-2}K_{00}^\sigma(\omega, \mathbf{q})Q_t(\omega) \end{aligned}$$

Tip-layer interaction:

$$V_{ts}(q) = \frac{4\pi e^2 e^{-q(d_t-d_s)}}{(1+\epsilon)q}$$



Results: detecting the Goldstone mode via microwave impedance microscopy

Correction to admittance

$$U_t(\omega) = Q_t(\omega)/C_t + \delta U(\omega)$$

$$Y_t(\omega) \equiv I_t(\omega)/U_t(\omega) = -i\omega Q_t(\omega)/U_t(\omega)$$

$$\text{Re}[\delta Y(\omega)] = \omega C_t^2 \mathcal{V}^{-1} \sum_{\sigma=\pm, \mathbf{q}} [g^\sigma(\mathbf{q})]^{-2} \text{Im}[K_{00}^\sigma(\omega, \mathbf{q})]$$

Low frequency: Goldstone mode contribution

$$g^-(\mathbf{q}) = \frac{4\pi e^{-qd_t}}{(1+\epsilon)q} [e^{-qd/2} - e^{qd/2}] \approx -\frac{4\pi d e^{-qd_t}}{(1+\epsilon)}$$

$$K_{00}^- = \frac{\kappa v^2 q^2}{v^2 q^2 - (\omega + i\eta)^2}$$

$$= \frac{\kappa v^2 q^2}{v^2 q^2 - \omega^2} + \frac{1}{2} i\pi \kappa v q [\delta(\omega - vq) - \delta(\omega + vq)]$$

κ : static interlayer capacitance (exciton compressibility)

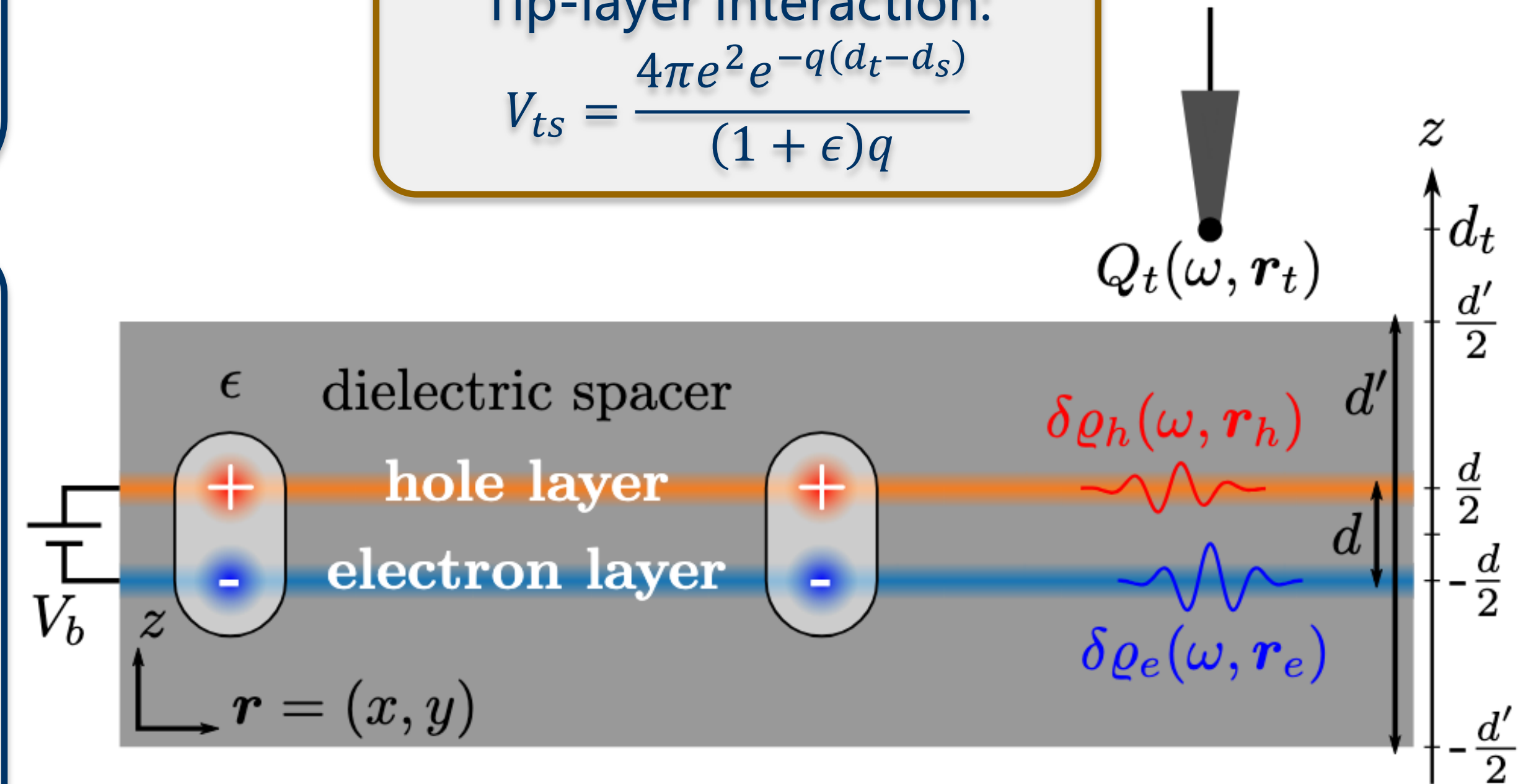
$$\text{Re}[\delta Y(\omega)] = \frac{\omega C_t^2 (4\pi d)^2}{(1+\epsilon)^2 \mathcal{V}} \sum_{\mathbf{q}} e^{-2qd_t} \frac{\pi \kappa v q}{2} \delta(\omega - vq)$$

$$= \frac{\omega C_t^2 (4\pi d)^2 \pi \kappa v}{2(1+\epsilon)^2} \int_0^\infty \frac{qdq}{2\pi} e^{-2qd_t} q \delta(\omega - vq)$$

$$= \frac{4\pi^2 d^2 C_t^2 \kappa}{(1+\epsilon)^2 v^2} \omega^3 e^{-2d_t \omega/v}$$

Tip-layer interaction:

$$V_{ts} = \frac{4\pi e^2 e^{-q(d_t-d_s)}}{(1+\epsilon)q}$$

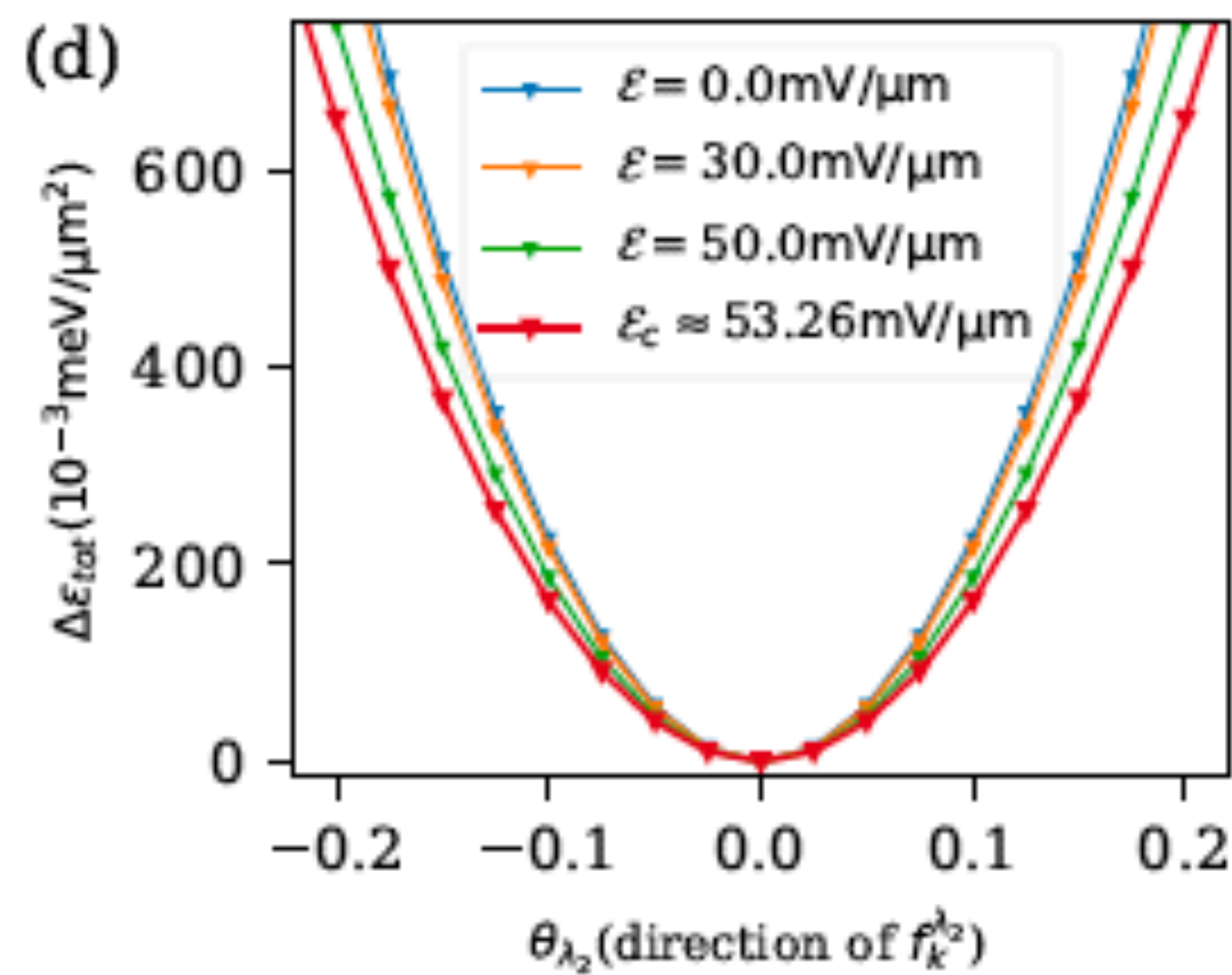
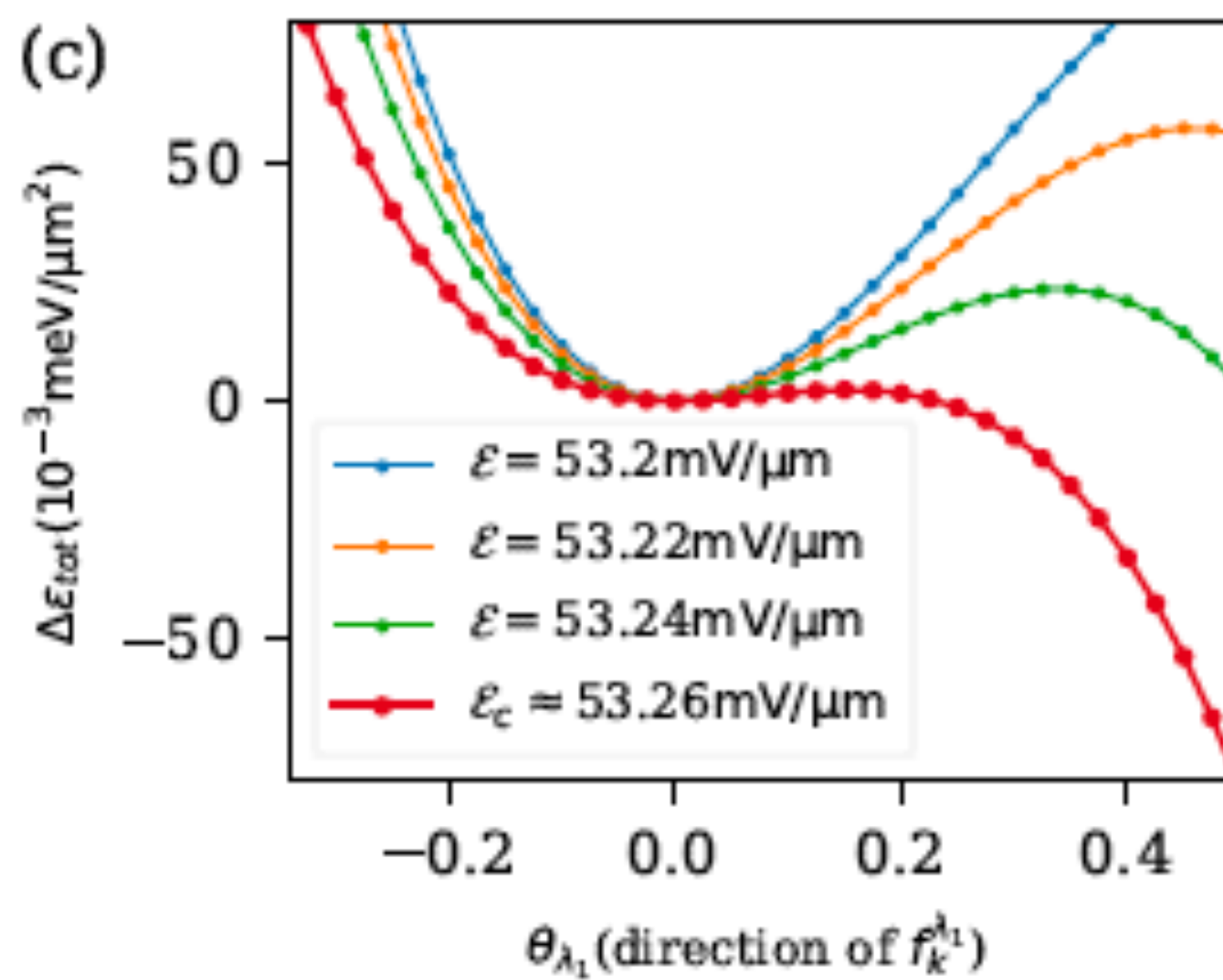
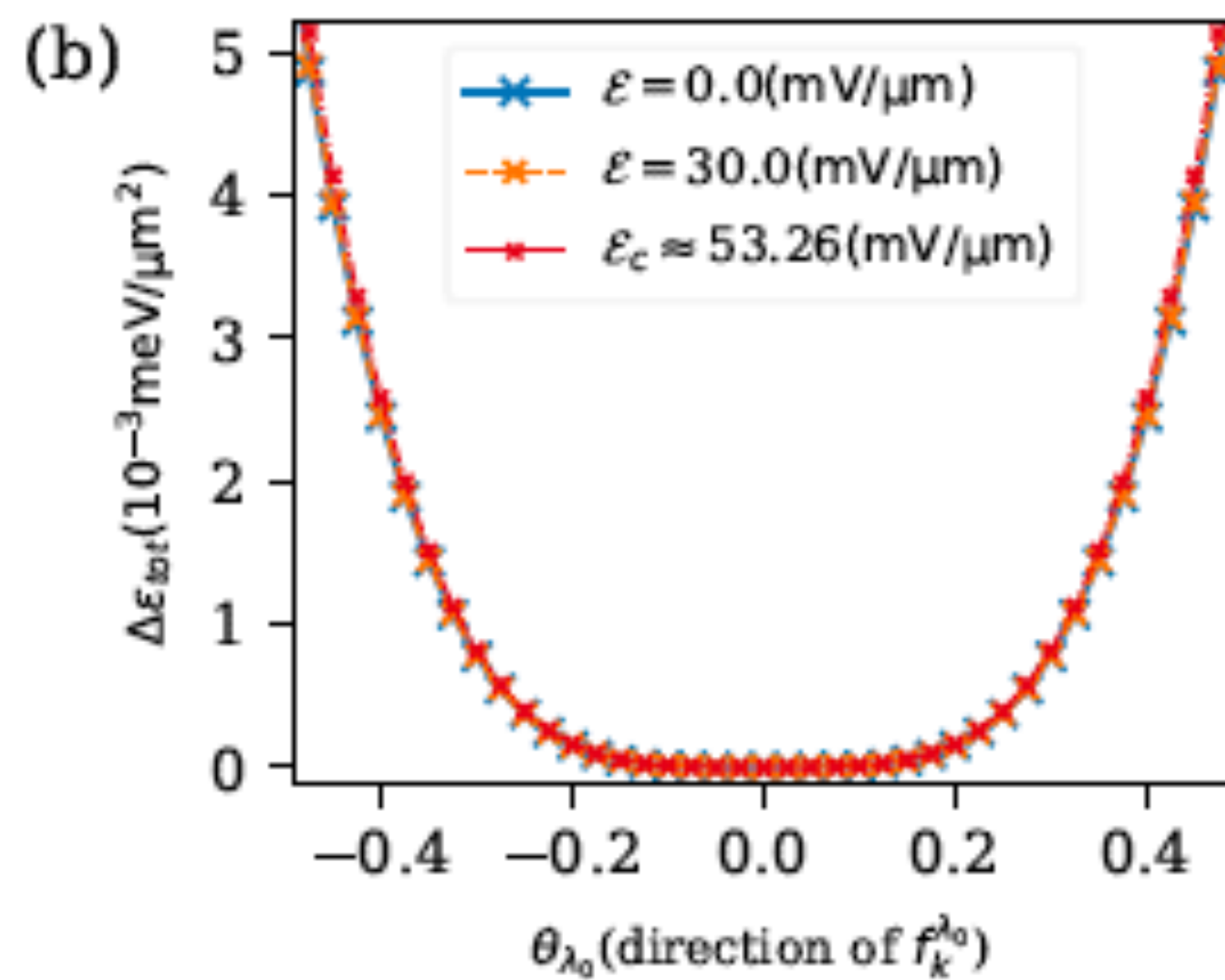
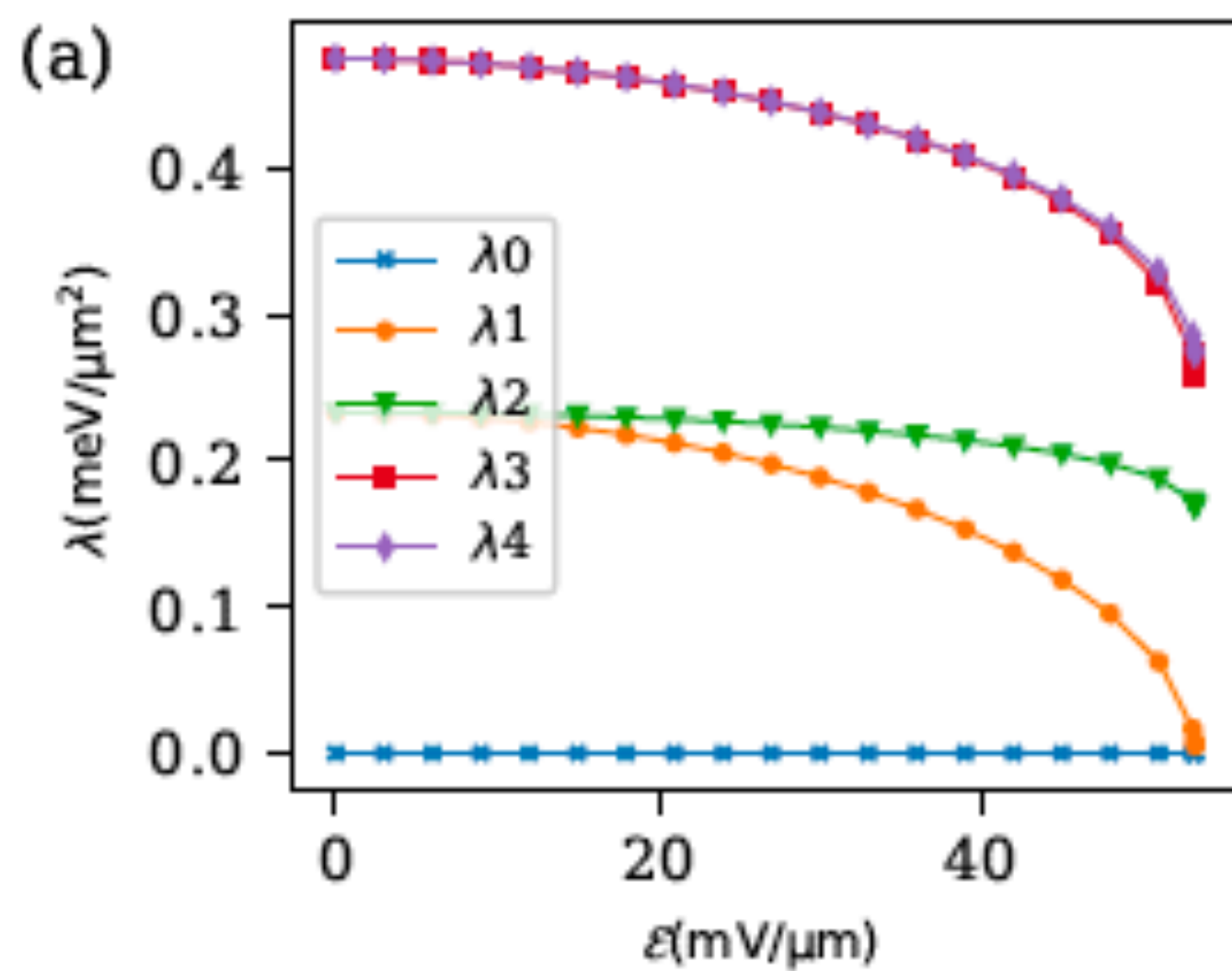


$$\text{Re}[\delta Y(\omega)] \sim \omega^3 e^{-\frac{2d_t \omega}{v}}$$

- Nonzero: Gapless excitations
- ω^3 dependence: Linear dispersion
- Exponential factor: Determine the velocity

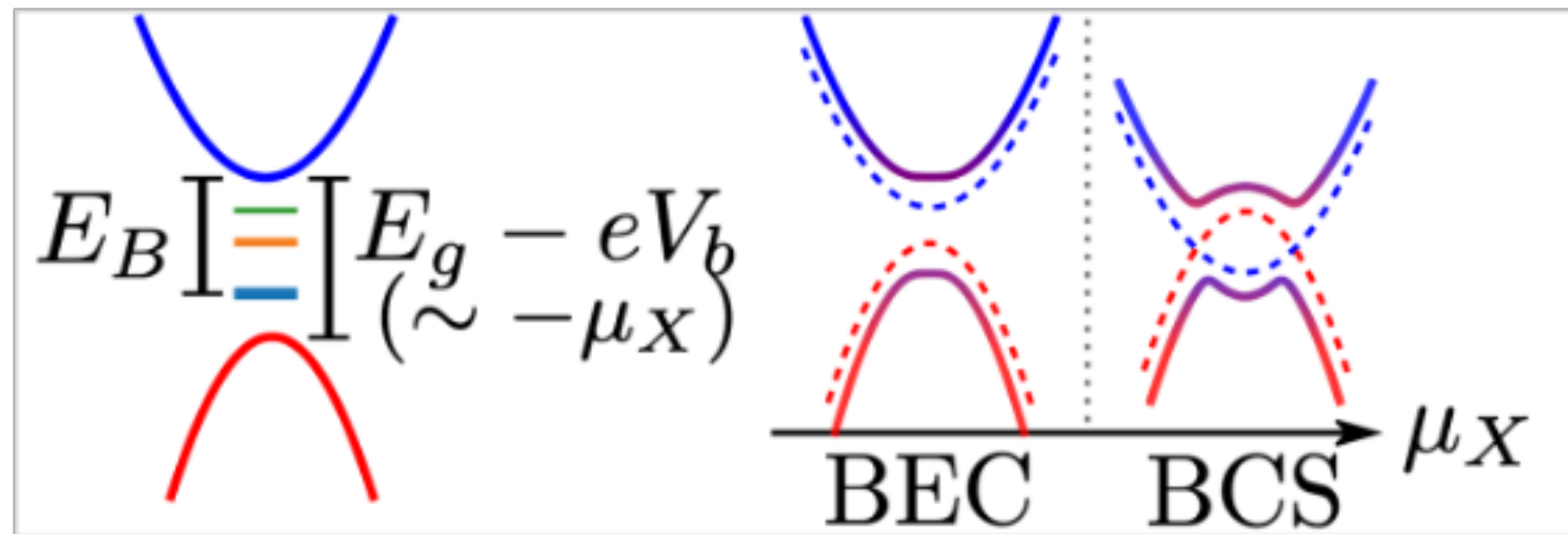
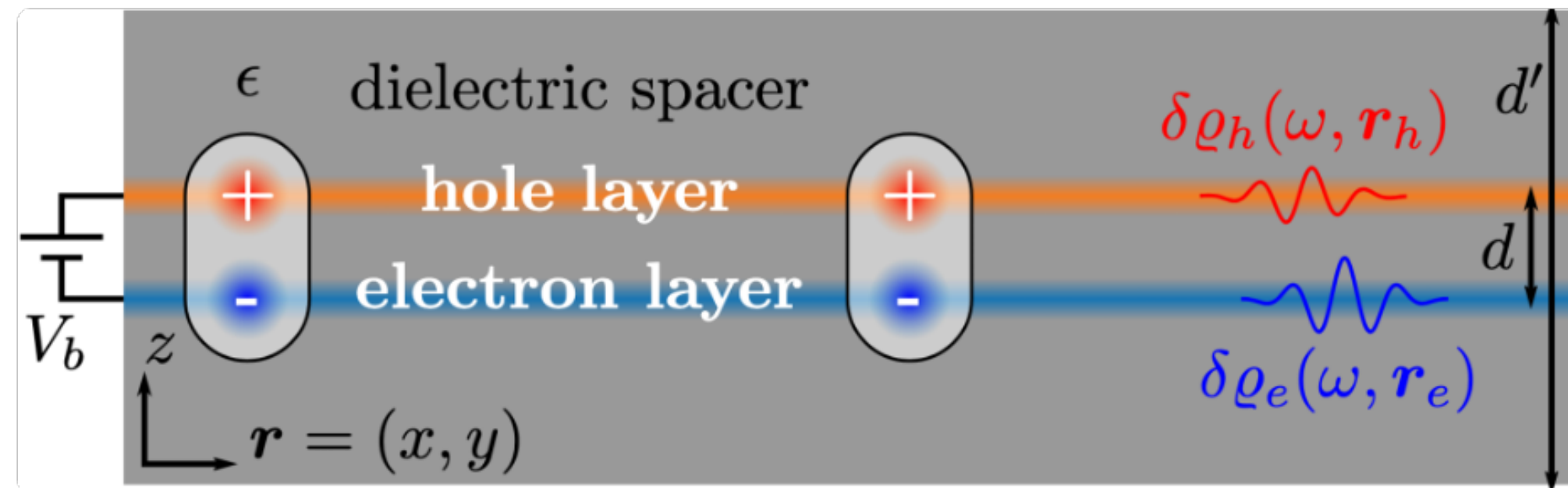
Summary

- Temperature dependence of gap: gap disappear above T_c *
- Electric break down: "pair breaking" type breakdown **
- Unusal Quantum capacitance **
- Perfect Coulomb drag ***
- quantum oscillation of insulators? **
- Meissner effect?
- More...



EM response of Inter-layer exciton systems

A new controllable platform to study EI



- Static E-field: Many-body Electric Breakdown by in-plane (PRX 14, 021047)
- Static B-field: quantum oscillation of quantum capacitance under out of plane B-field (<https://doi.org/10.1103/PhysRevB.109.155107>)
- Dynamic: what are the responses generated by different collect modes?
- Dynamic: what is the gauge field that couples to excitons?

Mean Field Hamiltonian and Self-Consistent Equation

- Minimum of the energy functional: $\delta \varepsilon_G / \delta \langle v\mathbf{k} | = 0$
- Mean field Hamiltonian $h_{\mathbf{k}}^{MF} = h_{\mathbf{k}}^0 + h^H + h_{\mathbf{k}}^F + h_{\mathbf{k}}^P$ where

$$h^H[|v\mathbf{k}\rangle] = \frac{4\pi e^2 n_{ex} d}{\epsilon} (1 - \rho^0),$$

$$h_{ss'\mathbf{k}}^F[|v\mathbf{k}\rangle] = -\frac{1}{\mathcal{V}} \sum_{\mathbf{k}'} V_{s's}(\mathbf{k} - \mathbf{k}') \tilde{\rho}_{ss'\mathbf{k}'},$$

$$h_{\mathbf{k}}^P[|v\mathbf{k}\rangle; \mathcal{E}] = \frac{ie\mathcal{E}}{2\Delta k_{\parallel}} \sum_{\sigma=\pm 1} \frac{\sigma |v\mathbf{k}_{\sigma}\rangle \langle v\mathbf{k}|}{\langle v\mathbf{k}|v\mathbf{k}_{\sigma}\rangle} + h.c..$$

- Self-consistent equation

$$h_{\mathbf{k}}^{MF}[|v\mathbf{k}\rangle; \mathcal{E}] |v\mathbf{k}\rangle = \xi_{v\mathbf{k}} |v\mathbf{k}\rangle$$